

ROBUST ENERGY TRANSFER MECHANISM VIA PRECESSION RESONANCE IN NONLINEAR TURBULENT WAVE SYSTEMS

Miguel D. Bustamante¹, Brenda Quinn² & Dan Lucas¹

¹*Complex and Adaptive Systems Laboratory, School of Mathematical Sciences, University College Dublin, Belfield, Dublin 4, Ireland*

²*School of Mechanical Engineering, Tel-Aviv University, Israel*

Abstract A robust energy transfer mechanism is found in nonlinear wave systems, which favours transfers towards modes interacting via triads with nonzero frequency mismatch, applicable in meteorology, nonlinear optics and plasma wave turbulence. We emphasise the concepts of truly dynamical degrees of freedom and triad precession. Transfer efficiency is maximal when the triads' precession frequencies resonate with the system's nonlinear frequencies, leading to a collective state of synchronised triads with strong turbulent cascades at intermediate nonlinearity. Numerical simulations confirm analytical predictions.

INTRODUCTION

We introduce a new robust mechanism of strong energy transfers in real physical systems, precisely in the context where the hypotheses of classical wave turbulence theory [6, 12, 10] do not hold, namely when the spatial domains have a finite size, when the amplitudes of the carrying fields are not infinitesimally small and when the linear wave timescales are comparable to the timescales of the nonlinear oscillations. The theory that deals with these energy exchanges is Discrete and Mesoscopic Wave Turbulence [11, 9, 7, 8, 4, 1, 2] and is still in development. Our results apply to a variety of systems, namely the nonlinear partial differential equations (PDEs) of classical turbulence, nonlinear optics, quantum fluids and magneto-hydrodynamics considered on bounded physical domains. For the sake of simplicity of presentation we discuss here the Charney-Hasegawa-Mima (CHM) equation [3, 5], a PDE governing Rossby waves in the atmosphere and drift waves in inhomogeneous plasmas: $(\nabla^2 - F)\frac{\partial\psi}{\partial t} + \beta\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x} = 0$, where in the plasma case the wave field $\psi(\mathbf{x}, t) \in \mathbb{R}$ is the electrostatic potential, $F^{-1/2}$ is the ion Larmor radius at the electron temperature and β is a constant proportional to the mean plasma density gradient.

We assume periodic boundary conditions: $\mathbf{x} \in [0, 2\pi)^2$. Decomposing the field in Fourier harmonics, $\psi(\mathbf{x}, t) = \sum_{\mathbf{k} \in \mathbb{Z}^2} A_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}}$ with wavevector $\mathbf{k} = (k_x, k_y)$, the components $A_{\mathbf{k}}(t)$, $\mathbf{k} \in \mathbb{Z}^2$ satisfy the evolution equation

$$\dot{A}_{\mathbf{k}} + i\omega_{\mathbf{k}} A_{\mathbf{k}} = \frac{1}{2} \sum_{\mathbf{k}_1, \mathbf{k}_2 \in \mathbb{Z}^2} Z_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} \delta_{\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}} A_{\mathbf{k}_1} A_{\mathbf{k}_2}, \quad (1)$$

where $Z_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} = (k_{1x}k_{2y} - k_{1y}k_{2x})\frac{|\mathbf{k}_1|^2 - |\mathbf{k}_2|^2}{|\mathbf{k}|^2 + F}$ are the interaction coefficients, $\omega_{\mathbf{k}} = \frac{-\beta k_x}{k^2 + F}$ are the linear frequencies and δ is the Kronecker symbol. Reality of ψ implies $A_{-\mathbf{k}} = A_{\mathbf{k}}^*$ (complex conjugate). Since the degree of nonlinearity in the PDE is quadratic, the modes $A_{\mathbf{k}}$ interact in triads. A triad is a group of any three spectral modes $A_{\mathbf{k}_1}(t)$, $A_{\mathbf{k}_2}(t)$, $A_{\mathbf{k}_3}(t)$ whose wavevectors satisfy $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$. The triad's *linear frequency mismatch* is defined by $\omega_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}_3} \equiv \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_3}$. Since any mode belongs to several triads, energy can be transferred nonlinearly throughout the intricate network or cluster of connected triads. In weakly nonlinear wave turbulence, triad interactions with non-zero frequency mismatch can be eliminated via a near-identity transformation. However, at finite nonlinearity these interactions cannot be eliminated *a priori* because they take part in the triad precession resonances presented below.

TRULY DYNAMICAL DEGREES OF FREEDOM AND PRECESSION RESONANCE

We introduce the amplitude-phase representation: $A_{\mathbf{k}} = \sqrt{n_{\mathbf{k}}} \exp(i\phi_{\mathbf{k}})$, where $n_{\mathbf{k}}$ is called the wave spectrum [10]. Energy $E = \sum_{\mathbf{k} \in \mathbb{Z}^2} (|\mathbf{k}|^2 + F)n_{\mathbf{k}}$ and enstrophy $\mathcal{E} = \sum_{\mathbf{k} \in \mathbb{Z}^2} |\mathbf{k}|^2 (|\mathbf{k}|^2 + F)n_{\mathbf{k}}$ are conserved at all times. In the context of CHM equation (Galerkin-truncated to N wavevectors), the truly dynamical degrees of freedom are any $N - 2$ linearly independent *triad* phases $\varphi_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}_3} \equiv \phi_{\mathbf{k}_1} + \phi_{\mathbf{k}_2} - \phi_{\mathbf{k}_3}$ [4] and the N wave spectrum variables $n_{\mathbf{k}}$. These $2N - 2$ degrees of freedom satisfy a closed system of evolution equations (individual phases $\phi_{\mathbf{k}}$ are obtained by quadrature):

$$\dot{n}_{\mathbf{k}} = \sum_{\mathbf{k}_1, \mathbf{k}_2} Z_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} \delta_{\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2} (n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2})^{\frac{1}{2}} \cos \varphi_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}}, \quad (2)$$

$$\dot{\varphi}_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}_3} = -\omega_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}_3} + \sin \varphi_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}_3} (n_{\mathbf{k}_3} n_{\mathbf{k}_1} n_{\mathbf{k}_2})^{\frac{1}{2}} \left[\frac{Z_{\mathbf{k}_2 \mathbf{k}_3}^{\mathbf{k}_1}}{n_{\mathbf{k}_1}} + \frac{Z_{\mathbf{k}_3 \mathbf{k}_1}^{\mathbf{k}_2}}{n_{\mathbf{k}_2}} - \frac{Z_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}_3}}{n_{\mathbf{k}_3}} \right] + \text{NNTT}_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}_3}, \quad (3)$$

where the second equation applies to any triad ($\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$). $\text{NNTT}_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}_3}$ is a short-hand notation for “nearest-neighbouring-triad terms”; these are nonlinear terms similar to the second term in the RHS of equation (3).

Precession resonance. The triad phases $\varphi_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}_3}$ have a subtle contribution to the energy of the system. Under plausible hypotheses, the RHS of Eq. (3) admits a zero-mode (in time): $\Omega_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}_3} \equiv \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \varphi_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}_3}(t') dt'$. This is by definition the *precession frequency* of the triad phase and is a nonlinear functional of the dynamical variables. Typically it does not perturb the energy dynamics because it is incommensurate with the frequency content of the nonlinear oscillations of the triad variables $\varphi_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}_3}$ and $n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, n_{\mathbf{k}_3}$.

However, in special circumstances a resonance occurs whereby the triad precession frequency $\Omega_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}_3}$ matches one of the typical nonlinear frequencies (generically denoted Γ) of the triad variables. In this case, the RHS of Eq.(2) will normally develop a zero-mode (in time), leading to a sustained growth of the energy in the corresponding wave spectrum $n_{\mathbf{k}}$, for some wavevector(s) \mathbf{k} . We call this a (nonlinear) triad precession resonance. When several triads are involved in this type of resonance, strong fluxes of enstrophy are exhibited through the network of interconnected triads, leading to coherent collective oscillations and cascades towards small scales.

Probing the strong transfer mechanism. This resonance is easily accessible via initial-condition uniform rescaling ($n_{\mathbf{k}} \rightarrow \alpha n_{\mathbf{k}}$), provided the linear frequency mismatch $\omega_{\mathbf{k}_1\mathbf{k}_2}^{\mathbf{k}_3}$ be nonzero for some triad. Figure 1 (left panels) show, for a low-dimensional model, that peaks in efficiency of enstrophy transfer are obtained at selected values of the initial-amplitude rescaling parameter α , coinciding with the values at which the precession resonance is hit and according to the theoretically-predicted values. Figure 1 (centre & right panels) show, for the full PDE model, that peaks in enstrophy transfer efficiency towards high wavenumbers are attained at intermediate amplitudes, corresponding to regimes when several triads enter into a collective precession resonance. Ultimately, the precession resonance is associated with the presence of periodic orbits and unstable manifolds in the phase space. More details are found in our PRL publication [2].

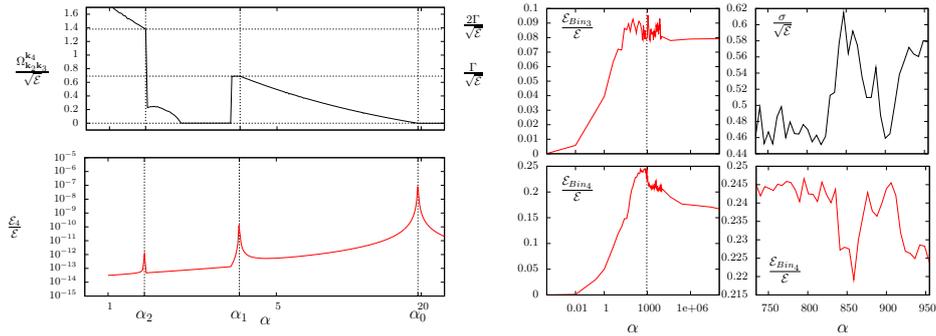


Figure 1. Left Panels: Numerical results from a low-dimensional model showing dimensionless precession (top) and enstrophy transfer efficiency to a target mode (bottom). Vertical lines indicate predicted resonances and show strong transfer efficiency at these values when precession resonance condition $\Omega_{23}^4 = p \Gamma$ is satisfied (horizontal lines, top). **Centre and Right Panels:** Numerical results from full PDE model at 128^2 resolution. Centre: enstrophy transfer efficiency against re-scaling factor α in high-wavenumber bins Bin_3 and Bin_4 . Vertical lines denote $\alpha = 900$. Right: dimensionless precession standard deviation (over all interacting triads) and enstrophy transfer efficiency in Bin_4 , both near efficiency peak $\alpha = 900$.

References

- [1] M. D. Bustamante and U. Hayat. Complete classification of discrete resonant Rossby/drift wave triads on periodic domains. *Commun. Nonlinear Sci. Numer. Simulat.*, **18**:2402–2419, 2013.
- [2] M. D. Bustamante, B. Quinn, and D. Lucas. Robust energy transfer mechanism via precession resonance in nonlinear turbulent wave systems. *Phys. Rev. Lett.*, **113**:084502, 2014.
- [3] J. G. Charney. Geostrophic turbulence. *Journal of the Atmospheric Sciences*, **28**(6):1087–1095, 1971.
- [4] K. Harper, M. D. Bustamante, and S. V. Nazarenko. Quadratic invariants for discrete clusters of weakly interacting waves. *J. Phys. A: Math. Theor.*, **46**:245501, 2013.
- [5] A. Hasegawa and K. Mima. Pseudo-three-dimensional turbulence in magnetised nonuniform plasma. *Phys. Fluids*, **21**:87–92, 1978.
- [6] K. Hasselmann. On the non-linear energy transfer in a gravity-wave spectrum Part 1. General theory. *J. Fluid Mech.*, **12**:481–500, 1962.
- [7] E. Kartashova. Exact and quasiresonances in discrete water wave turbulence. *Phys. Rev. Lett.*, **98**:214502, 2007.
- [8] V. S. L’Vov and S. V. Nazarenko. Discrete and mesoscopic regimes of finite-size wave turbulence. *Phys. Rev. E*, **82**:056322–1, 2010.
- [9] S. V. Nazarenko. Sandpile behavior in discrete water-wave turbulence. *J. Stat. Mech. Theor. Exp.*, page L02002, 2006.
- [10] S. V. Nazarenko. *Wave Turbulence (Lecture notes in Physics)*. Springer, 2011.
- [11] V. E. Zakharov, A. O. Korotkevich, A. N. Pushkarev, and A. I. Dyachenko. Mesoscopic wave turbulence. *JETP Letters*, **82**:487–491, 2005.
- [12] V. S. Zakharov, V. S. Lvov, and G. Falkovich. *Kolmogorov Spectra of Turbulence*. Springer-Verlag, Berlin, 1992.