THE ROLE OF HIGH VORTICITY STRUCTURES IN DEVELOPMENT OF KOLMOGOROV TURBULENT SPECTRA IN INVISCID FLOW

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Abstract This work is aimed for understanding nonlinear mechanisms at early stages of turbulence, when the flow is not yet affected by viscosity. Based on numerical simulations of the 3D incompressible Euler equations with generic large-scale initial conditions, we show that the exponential growth of vorticity developing in thin vortex sheets (pancake structures) leads to formation of Kolmogorov energy spectrum $E_k \propto k^{-5/3}$ in fully inviscid flow. This direct observation yields the decoupling of the finite-time blowup problem from the Kolmogorov theory of turbulence. We demonstrate that the pancake structures have self-similar dynamics and evolve according to the scaling law $\omega_{\max}(t) \propto \ell(t)^{-2/3}$ for the local vorticity maximums $\omega_{\max}(t)$ and the transverse pancake scales $\ell(t)$. Then, we argue that the energy spectrum requires an increasing number of such structures developing densely through the Kolmogorov range of wavenumbers, in good agreement with numerical data.

NUMERICAL SET-UP

Our numerical work is aimed for understanding nonlinear mechanisms at early stages of turbulence in relation to the long-standing blowup problem for 3D incompressible Euler equations [1] and the Kolmogorov theory of developed turbulence [2]. We solve the incompressible 3D Euler equations in the vorticity formulation,

$$\partial_t \boldsymbol{\omega} = \mathrm{rot}\,(\mathbf{v} \times \boldsymbol{\omega}), \quad \mathbf{v} = \mathrm{rot}^{-1}\boldsymbol{\omega}, \tag{1}$$

in the periodic box $\mathbf{r} = (x, y, z) \in [-\pi, \pi]^3$ using the pseudo-spectral numerical method with high-order Fourier filtering [3]. Assuming the vanishing average velocity $\int \mathbf{v} d^3\mathbf{r} = 0$, the inverse of the rotor operator in Eq. (1) is uniquely defined. We use adaptive rectangular grid with the number of nodes adapted independently along each coordinate. Our aim is the systematic study of the developing high vorticity structures, including the region of global vorticity maximum, and their relation to the energy spectrum. We tested several large-scale initial conditions in the form of random truncated Fourier series. This publication is based on one selected simulation, which demonstrated the longest Kolmogorov interval with the final grid $486 \times 1024 \times 2048$. We also tested other initial conditions and obtained similar results [4].

STRUCTURE OF HIGH VORTICITY REGIONS

In our simulation we observe that the number of high vorticity regions increases with time. These regions can be analyzed by looking at local maximums of the vorticity modulus $|\boldsymbol{\omega}|$. The vorticity growth at these maximums tends to be exponential $\omega_{\max} \propto e^{t/T_\omega}$ with close characteristic times T_ω. The global vorticity maximum grows with time from the initial value $\omega_{\max}(0) \approx 1.5$ to 11.8 at the finial simulation time $t = 6.89$. The regions of enhanced vorticity represent in space very thin vortex sheets (pancakes). The thicknesses of these pancakes ℓ nearly exponentially decrease in time $\ell \propto e^{-t/T_\ell}$ with close characteristic times T_ℓ, while the longitudinal scales ξ remain almost the same, $\xi \sim 1$. We observe a clear tendency of local maximums to follow asymptotically the power-law relation between the maximum vorticity and pancake thickness, Fig. 1(a),

$$\omega_{\max}(t) \propto \ell(t)^{-2/3}. \tag{2}$$

It is remarkable that all local maximums follow approximately the same scaling law (2), with some shared constant prefactor before $\ell^{-2/3}$. A self-similar pancake model was suggested in [5] with the vorticity inverse proportional to the pancake width. This model is not compatible with scaling (2). We propose the modified asymptotic model [4] which satisfies the Euler equations (1) for the leading-order terms $\propto \omega_{\max}(t)$, while the next-order terms $\propto \omega_{\max}(t)\ell(t)$ exponentially decay in time for the scaling (2). This model demonstrates fairly well agreement with the simulation results.

DEVELOPMENT OF KOLMOGOROV ENERGY SPECTRUM

Now let us consider the evolution of the energy spectrum $E_k(t) = \frac{1}{2}\int |\mathbf{v}(\mathbf{k}, t)|^2\, k^2 do$. We observe that this spectrum has the "energy containing" region from which the energy flows into the small scales, and a self-similar tail at large

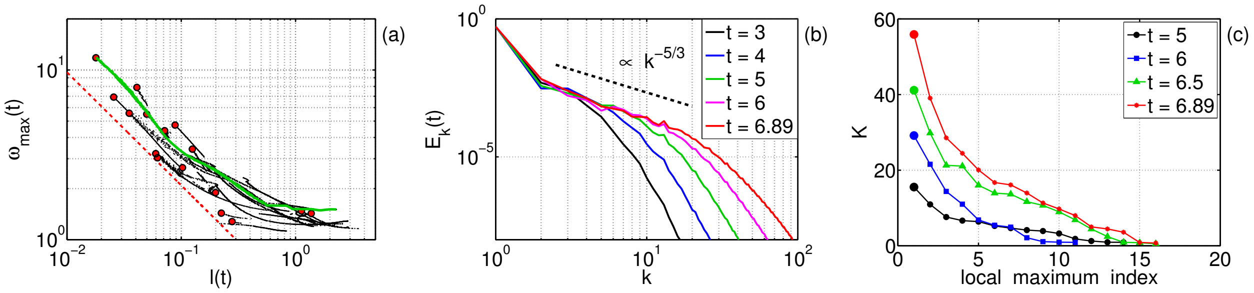

Figure 1. *(Color on-line)* (a) Vorticity local maximums $\omega_{\max}(t)$ vs. the respective characteristic lengths $\ell(t)$ during the evolution of the pancake structures. Green line shows the global maximum, red circles mark local maximums at the final time $t = 6.89$. Dashed red line indicates the power-law $\omega_{\max} \propto \ell^{-2/3}$. (b) Energy spectrum at different times demonstrating the Kolmogorov power-law. (c) Characteristic wavenumbers $K = 1/\ell$ in decreasing order for all local vorticity maximums vs. maximum index numbers at different times. Large circles mark the global maximums.

wavenumbers that evolves close to $E_k(t) \propto e^{-\delta(t)k}$. The exponent $\delta(t)$ behaves exponentially in time, $\delta(t) \propto e^{-t/T_\delta}$, see e.g. [5]. Between these two regions we observe the gradual formation of the Kolmogorov interval $E_k \propto k^{-5/3}$, Fig. 1(b). At the end of the simulation this interval extends to a decade of wavenumbers, $2 \leq k \leq 20$. The Kolmogorov interval is characterized by the "frozen" part of the spectrum in contrast to the vast changes with time at larger wavenumbers. To understand the specific mechanisms leading to formation of the Kolmogorov spectrum, we first estimate the contribution of the high vorticity regions. In Fourier space, each pancake generates a "jet" structure extended in one direction and aligned with the pancake normal vector. Inside such a jet, occupying only a small fraction of the Fourier space, the Fourier components of the flow are large in comparison with the remaining background. We compare the contributions to the energy spectrum from these jets and from the background, and demonstrate that the energy spectrum is determined by the collection of the pancake structures [4].

Then, we estimate the contribution of the asymptotic self-similar pancake model that we developed in the previous Section. A pancake structure influences significantly to the region of wavenumbers $|k| \leq 1/\ell = K$. The distribution of characteristic wavenumbers K for the local maximums in decreasing order is shown in Fig. 1(c). Most of the local maximums fill densely the Kolmogorov interval of wavenumbers $2 \leq k \leq 20$, however several of the local maximums, including the global maximum, have characteristic wavenumbers significantly exceeding the Kolmogorov region. This leads us to supposition that the Kolmogorov energy spectrum $E_k \propto k^{-5/3}$ is determined not by the inner part of the pancake structures (the "crests"), but by how these structures are connected to the background (the "tails"). However, under this supposition the asymptotic self-similar pancake model with the scaling (2) yields the different energy spectrum $E_p(k) \propto k^{-8/3}$ for the pancake structures [4]. Then, we argue that the energy cascade must be driven by a multitude of the pancake structures, those number should increase with the size of the Kolmogorov interval. The new pancakes may "feed" the energy from the tails of the "old" ones, thus enhancing the energy cascade to small scales. This idea reproduces the famous Richardson's notion of turbulence [6] as a continuous process of formation of eddies of different sizes, where the "eddies" take the form of pancakes in our case. Such mechanism of the energy cascade agrees with our numerical results: the Kolmogorov interval in Fig. 1(b) belongs to the interval of wavenumbers, where the pancakes appear densely with nearly a constant slope in Fig. 1(c). We conclude that the further study in this direction is necessary.

References

[1] J. Gibbon. The three-dimensional Euler equations: Where do we stand? *Physica D: Nonlinear Phenomena*, **237**(14):1894–1904, 2008.
[2] L. Landau, E. Lifshitz. Fluid Mechanics. Course of Theoretical Physics, **6**. Elsevier, 2013.
[3] T. Hou, R. Li. Computing nearly singular solutions using pseudo-spectral methods. *J. Comp. Phys.*, **226**(1):379–397, 2007.
[4] D. Agafontsev, E. Kuznetsov, A. Mailybaev. Development of high vorticity structures in incompressible 3D Euler equations, in preparation.
[5] M. Brachet, M. Meneguzzi, A. Vincent, H. Politano, and P. Sulem. Numerical evidence of smooth self-similar dynamics and possibility of subsequent collapse for three-dimensional ideal flows. *Phys. Fluids A*, **4**:2845–2854, 1992.
[6] L. Richardson. Weather Prediction by Numerical Process. Cambridge University Press, 2007.