A UNIFIED SHELL MODEL FOR BUOYANCY-DRIVEN TURBULENCE

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Abstract We construct a unified shell model for stably stratified and convective turbulence. Shell model simulation of stably stratified flow in turbulent regime exhibit Bolgiano-Obukhbov (BO) scaling in which the kinetic energy spectrum varies as $k^{-11/5}$. However, simulation of convective turbulence shows Kolmogorov's spectrum. These results are consistent with the direct numerical simulations of Kumar *et al.* [Phys. Rev. E **90**, 023016 (2014)]. We also observe a dual scaling ($k^{-11/5}$ and $k^{-5/3}$) for a limited range of parameters in stably stratified flow.

Buoyancy-driven flows [1] can be categories as: (a) Stably stratified flows in which the hotter fluid is above the colder fluid. These flows are stable; (b) Convective flows in which the colder fluid is above the hotter fluid. Such flows are unstable, since, the heavier fluid comes down, and the lighter fluid goes up with gravity acting downward. In this paper, we will discuss the turbulent aspects of such flows.

Direct numerical simulation (DNS) of turbulent flows with a moderate Reynolds number (Re) is quite difficult. For example, DNS of flows with $\mathrm{Re} \approx 10^6$ requires about trillion grid points [2], which is impossible even for the best of present supercomputer. In a shell model, flows with large Reynolds number can be easily simulated (approximately) with 40 or more shell variables, with each variable representing all the modes of the corresponding logarithmically-binned shell [3, 4]. Thus, a shell model helps us study turbulent flow. In this paper we present a study of turbulent buoyancy-driven flow using a unified shell model.

Shell Model: Our shell model [5] for the buoyancy-driven turbulence is

$$\frac{du_n}{dt} = N_n[u,u] + \alpha g \theta_n - \nu k_n^2 u_n + f_n, \tag{1}$$

$$\frac{d\theta_n}{dt} = N_n[u,\theta] - \frac{d\bar{T}}{dz} u_n - \kappa k_n^2 \theta_n, \tag{2}$$

where u_n and θ_n are the shell variables for the velocity and temperature fluctuations respectively, f_n represents the external force field, $k_n = k_0 \lambda^n$ is the wavenumber of the n-th shell, and ν and κ are fluid's kinematic viscosity and thermal diffusivity respectively. Here $\alpha g \theta_n$ and $d\bar{T}/dz$ are the buoyancy and temperature gradient respectively, where α is the thermal expansion coefficient, and g is the acceleration due to gravity. Note that $d\bar{T}/dz > 0$ for the stably stratified flow, but $d\bar{T}/dz < 0$ for the convective turbulence. We use Sabra model [6] to construct the nonlinear terms $N_n[u,u]$ and $N_n[u,\theta]$ as

$$N_n[u,u] = -i(a_1 k_n u^*_{n+1} u_{n+2} + a_2 k_{n-1} u^*_{n-1} u_{n+1} - a_3 k_{n-2} u_{n-1} u_{n-2}) \tag{3}$$

$$\begin{aligned} N_n[u,\theta] = {} & -i[k_n(d_1 u^*_{n+1}\theta_{n+2} + d_3 \theta^*_{n+1} u_{n+2}) + k_{n-1}(d_2 u^*_{n-1}\theta_{n+1} - d_3 \theta^*_{n-1} u_{n+1}) \\ & -k_{n-2}(-d_1 u_{n-1}\theta_{n-2} - d_2 \theta_{n-1} u_{n-2})] \end{aligned} \tag{4}$$

where $a_1 = d_1 = 1$, $a_2 = d_2 = \lambda - 2$, and $a_3 = d_3 = 1 - \lambda$, where $\lambda = (\sqrt{5}+1)/2$ is the golden mean [4]. The boundary conditions are $u_{-1} = u_0 = u_{M+1} = u_{M+2} = 0$ and $\theta_{-1} = \theta_0 = \theta_{M+1} = \theta_{M+2} = 0$, where M is the total number of shells. In absence of the external forcing, the buoyancy, and the stratification, the above model conserves [4] the kinetic energy $\sum_n |u_n|^2/2$, the kinetic helicity $\sum_n (-1)^n k_n |u_n|^2$, and the entropy $\sum_n |\theta_n|^2/2$. The important nondimenesional parameters are the Prandtl number $\mathrm{Pr} = \nu/\kappa$ and the Rayleigh number $\mathrm{Ra} = (\alpha g d^4/\nu\kappa)(d\bar{T}/dz)$.

Results: For stably stratified turbulence, we simulate the shell model for $\mathrm{Pr} = 1$, $\mathrm{Ra} = 10^5$, and the energy supply rate of 50. We obtain the Froude number $\mathrm{Fr} = 3.2$ and $\mathrm{Re} = 10^3$. The Froude number is the ratio of the characteristic velocity and the gravitational wave velocity. In Fig. 1(a) we plot the kinetic energy (KE) and entropy spectra that exhibits the Bolgiano-Obukhbov [7, 8] scaling. To obtain dual scaling, we need a higher range of wavenumber for which we increase the Rayleigh number to 10^{10}. Consequently we obtain $\mathrm{Re} = 2.0 \times 10^5$ and $\mathrm{Fr} = 2.0$. We observe dual scaling for the kinetic energy spectrum, as shown in Fig. 1(b). The wavenumber range $4 < k < 18$ exhibit Bolgiano scaling [$E_u(k) \sim k^{-11/5}$], and $18 < k < 100$ exhibit Kolmogorov scaling [$E_u(k) \sim k^{-5/3}$].

For convective turbulence, we performed simulation for $\mathrm{Pr} = 1$ and $\mathrm{Ra} = 10^{12}$, which yields, $\mathrm{Re} = 8.7 \times 10^6$. In Fig. 2(a) we plot the KE and entropy spectra which shows Kolmogorov scaling, i.e. $E_u(k) \sim k^{-5/3}$ and $E_\theta(k) \sim k^{-5/3}$. As shown in Fig. 2(b), we obtain constant energy fluxes in inertial range ($20 < k < 1000$), consistent with the Kolmogorov scaling.

In summary, we present a unified shell model for buoyancy-driven turbulence which is applicable to both stably stratified and convective turbulence. Earlier, Brandenburg [9], Mingshul and Shida [10], and Ching and Cheng [11] had constructed shell models for convective turbulence, but their results were divergent, and their shell models are not fully

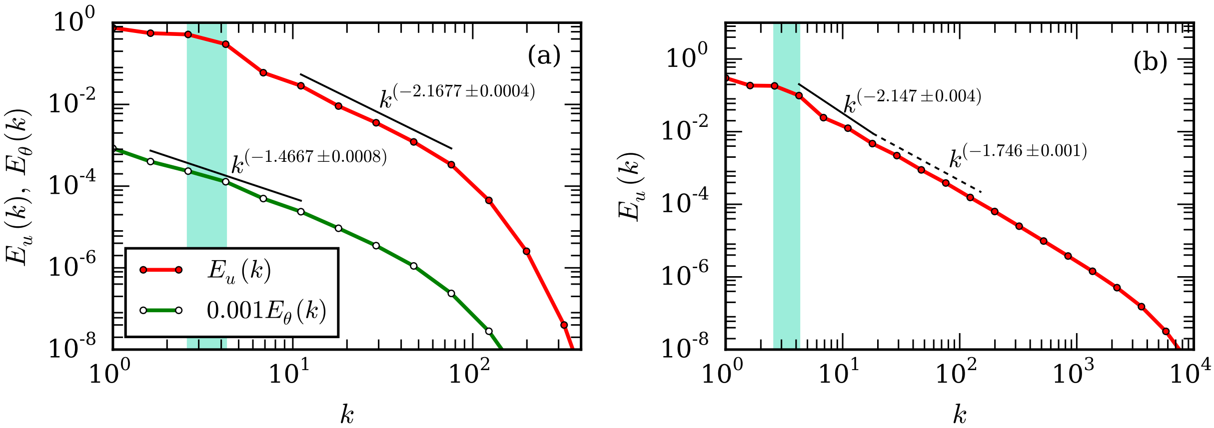

Figure 1. For stably stratified simulation: (a) plots of KE and entropy spectra for $\mathrm{Pr} = 1$, $\mathrm{Ra} = 10^5$, and $\mathrm{Ri} = 0.10$ and (b) plot of KE spectrum for $\mathrm{Pr} = 1$, $\mathrm{Ra} = 10^{10}$, and $\mathrm{Ri} = 0.25$. In Fig. (b), the wavenumber range $4 < k < 18$ exhibit Bolgiano scaling, and $18 < k < 100$ exhibit Kolmogorov scaling (thus dual scaling). The green shaded region shows the forcing range.

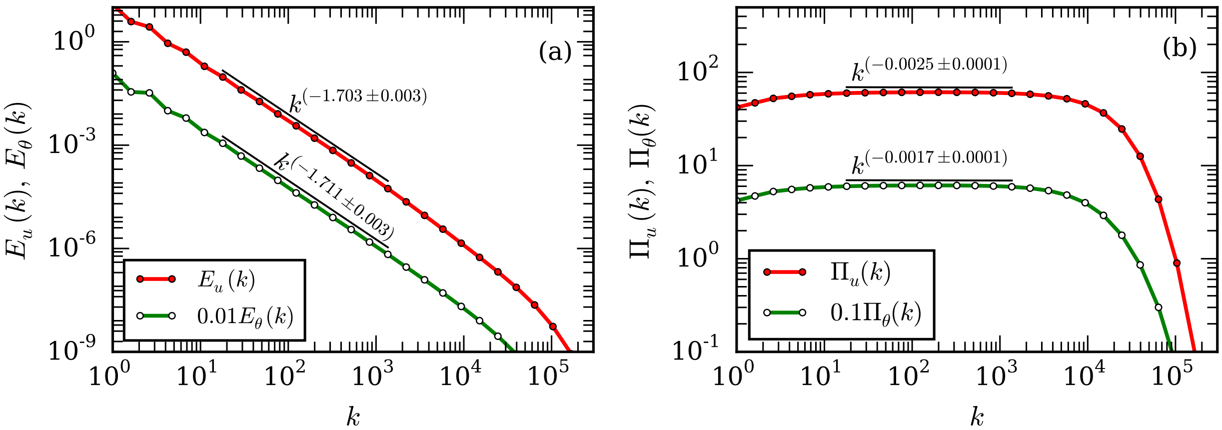

Figure 2. For convective turbulence simulation with $\mathrm{Pr} = 1$ and $\mathrm{Ra} = 10^{12}$: (a) plots of KE and entropy spectra; (b) plots of KE flux $\Pi_u(k)$ and entropy flux $\Pi_\theta(k)$.

satisfactory. Our model overcomes these deficiencies and yields BO scaling for stably stratified turbulence when buoyancy is significantly strong. We also obtain dual spectrum for stably stratified flows for a narrow set of parameters. Note that very strong buoyancy ($\mathrm{F}r \ll 1$) produces quasi two-dimensional flow [12, 13], but the parameter regime of our shell model differs from this regime. Our shell model for the convective turbulence exhibits Kolmogorov scaling. The results from our shell model are consistent with those of DNS of Kumar *et al.* [14].

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