SWEEPING HAS NO EFFECT ON RENORMALIZED TURBULENT VISCOSITY

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Abstract We perform renormalization group analysis (RG) of the Navier-Stokes equation in the presence of constant mean velocity field $\mathbf{U}_0$, and show that the renormalized viscosity is unaffected by $\mathbf{U}_0$, thus negating the "sweeping effect", proposed by Kraichnan [Phys. Fluids **7**, 1723 (1964)] using random Galilean invariance. Using direct numerical simulation, we show that the correlation functions $\langle \mathbf{u}(\mathbf{k},t)\mathbf{u}(\mathbf{k},t+\tau)\rangle$ for $\mathbf{U}_0 = 0$ and $\mathbf{U}_0 \neq 0$ differ from each other, but the renormalized viscosity for the two cases are the same. Our numerical results are consistent with the RG calculations.

Navier Stokes equation satisfies the Galilean invariance. Kraichnan [1], however, argued that the "random" Galilean invariance is not obeyed in Eulerian formulation of fluid flow due to the *sweeping effect*, according to which small-scale fluid structures are convected by the large energy-containing eddies. Kraichnan considered a fluid flow with a *random* mean velocity field, which is constant in space and time, but has a Gaussian and isotropic distribution over an ensemble of realisations. An application of direct interaction approximation to such system yields $E(k) \sim (\Pi U_{0,\mathrm{rms}})^{1/2} k^{-3/2}$, where $U_{0,\mathrm{rms}}$ is the rms value of the mean velocity. This observation is contrary to the observations ($E(k) \sim k^{-5/3}$), therefore Kraichnan claimed inadequacies of the Eulerian formalism for obtaining Kolmogorov's spectrum for fully developed fluid turbulence. Later, he developed Lagrangian field theory of fluid turbulence that is consistent with the Kolmogorov's 5/3 theory of turbulence.

We performed renromalization group analysis of the Navier-Stokes equation in the presence of a "constant" mean velocity field $\mathbf{U}_0$ using the methods proposed by McComb, Zhou, and coworkers [2, 3]. We show that the renormalized viscosity is independent of the mean velocity, thus showing an absence of the sweeping effect on the renormalized viscosity. The energy spectrum is independent of the mean velocity, and it follows Kolmogorov's spectrum.

In this renormalization process, the wavenumber range (k_N, k_0) is divided logarithmically into N shells. The nth shell is (k_n, k_{n-1}) where $k_n = h^n k_0$ $(h < 1)$. In the first step, the spectral space is divided in two parts: the shell $(k_1, k_0) = k^>$, which is to be eliminated, and $(k_N, k_1) = k^<$, set of modes to be retained. The equation for a Fourier modes belonging to $k^<$ is

$$[-i\omega(k) + i\mathbf{U}_0\cdot\mathbf{k} + (\nu_0(k) + \delta\nu_0(k))k^2]u_i^<(\hat{k}) = -\frac{i}{2}P_{ijm}(\mathbf{k})\int d\mathbf{p}d\omega(p)[u_j^<(\hat{p})u_m^<(\hat{k}-\hat{p})] + f_i(\hat{k}) \tag{1}$$

where the correction to the kinematic viscosity $\delta\nu_0(k)$ is given by

$$\begin{aligned}
\delta\nu_0(k)k^2 &= \frac{1}{2}\int_{\hat{p}+\hat{q}=\hat{k}}^{\Delta} d\mathbf{p}d\omega(p)[S(k,p,q)G(\hat{q})C(\hat{p})] \\
&= \frac{1}{2}\int_{\hat{p}+\hat{q}=\hat{k}}^{\Delta} d\mathbf{p}d\omega(p)\frac{S(k,p,q)C(p)}{(-i\omega(q)+i\mathbf{U}_0\cdot\mathbf{q}+\nu_0(q)q^2)(-i\omega(p)+i\mathbf{U}_0\cdot\mathbf{p}+\nu_0(p)p^2)} \\
&= \frac{1}{2}\int_{\hat{p}+\hat{q}=\hat{k}}^{\Delta} d\mathbf{p}\frac{S(k,p,q)C(\mathbf{p})}{[-\nu_0(k)k^2+\nu_0(p)p^2+\nu_0(q)q^2+\cancel{(-i\mathbf{U}_0\cdot\mathbf{k}+i\mathbf{U}_0\cdot\mathbf{p}+i\mathbf{U}_0\cdot\mathbf{q})}]} \\
&= \frac{1}{2}\int_{\hat{p}+\hat{q}=\hat{k}}^{\Delta} d\mathbf{p}\frac{S(k,p,q)C(\mathbf{p})}{-\nu_0(k)k^2+\nu_0(p)p^2+\nu_0(q)q^2}.
\end{aligned} \tag{2}$$

where $G(\hat{q})$ is Green's function, $C(\hat{p}) = C(\mathbf{p})/(-i\omega(p)+i\mathbf{U}_0\cdot\mathbf{p}+\nu_0(p)p^2)$ is the correlation function, and $S(k,p,q) = kp((d-3)z + 2z^3 + 2xy)$, where d is the space dimension. The integral is performed over the first shell $\Delta = (k_1, k_0)$. In the above computation we also use $\omega(k) = \omega(p) + \omega(q)$ and $i\omega(k) = i\mathbf{U}_0\cdot\mathbf{k} + \nu_0(k)k^2$.

Our computation clearly shows that the correction to the kinematic viscosity is independent of $\mathbf{U}_0$. As a result, the renormalized viscosity after the first step is $\nu_1(k) = \nu_0(k) + \delta\nu_0(k)$, and subsequent steps are also independent of $\mathbf{U}_0$. Following the subsequent steps of McComb [2, 4, 5] and Zhou [3, 6] for $\mathbf{U}_0$, we can show that the

$$E(k) = K_{Ko}\Pi^{2/3}k^{-5/3} \tag{3}$$

$$\nu(k) = \nu_*\sqrt{K_{Ko}}\Pi^{1/3}k^{-4/3} \tag{4}$$

where $E(\mathbf{k}) = 4\pi k^2 C(k)$, K_{Ko} is the Kolmogorov's constant, Π is the energy flux, $\nu(k)$ is the renormalized viscosity, and ν_* is a constant that satisfies the RG equations. In 3D, $\nu_* \approx 0.4$. These results are same as those obtained for $\mathbf{U}_0 = 0$. We compute the renormalized viscosity using direct numerical simulations [7] for $\mathbf{U}_0 = 0$ and $\mathbf{U}_0 = 10$. We employ the following formula [8] for the same:

$$R(\mathbf{k},\tau) = \frac{C(\mathbf{k},\tau)}{C(\mathbf{k},0)} = \frac{\langle \mathbf{u}(\mathbf{k},t)\cdot\mathbf{u}^*(\mathbf{k},t+\tau)\rangle}{\langle|\mathbf{u}(\mathbf{k},t)|^2\rangle}$$

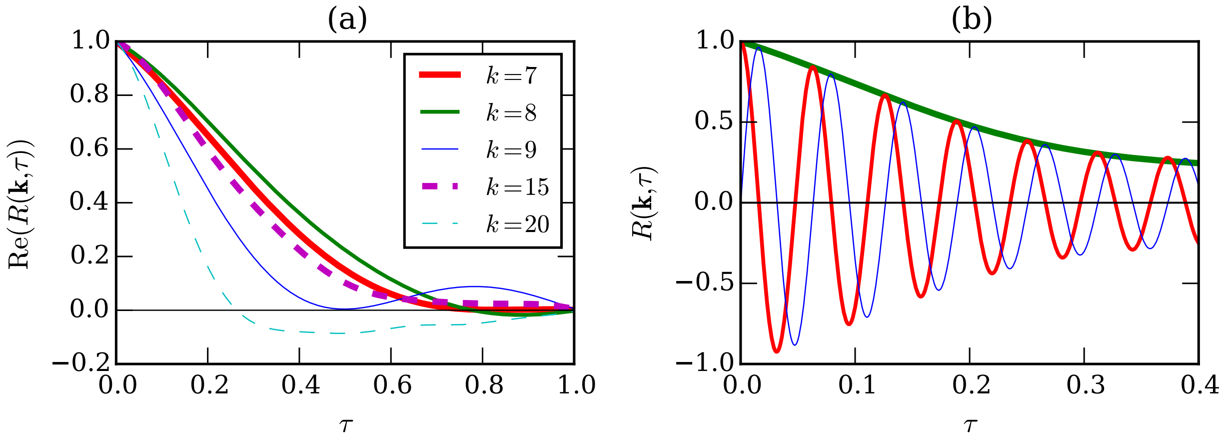

Figure 1. (a) Plots of the normalised correlation function $\mathrm{Re}(R(\mathbf{k},\tau))$ for $k = 7, 8, 9, 15, 20$; (b) For $U_0 = 10$ and $\mathbf{k} = (0, 0, 10)$, $\mathrm{Re}(R(\mathbf{k},\tau))$ (thick red) and $\mathrm{Im}(R(\mathbf{k},\tau))$ (thin blue), exhibit damped oscillations. $\mathrm{Re}(R(\mathbf{k},\tau))$ (thick green) for $U_0 = 0$ envelopes $R(\mathbf{k},\tau)$ for $U_0 = 10$, thus demonstrating the $\nu(k)$ is same for $U_0 = 0$ and 10.

(5)

For $\mathbf{U}_0 = 0$, the function $\mathrm{Re}(R(\mathbf{k},\tau))$ for $\mathbf{k} = (0, 0, k)$ with $k = 7, 8, 9, 15, 20$ are plotted in Fig. 1(a). We observe that the normalised correlation function decays with a timescale $\tau_c(k) = \nu(k)k^2$, thus validating the renormalization procedure described above. For $\mathbf{U}_0 = 10$, the function $\mathrm{Re}(R(\mathbf{k},\tau))$ for $\mathbf{k} = (0, 0, 10)$ is shown in Fig. 1(b). The normalised correlation function exhibits damped oscillations with $\omega = k_z U_0$ and decay time scale of $1/(\nu(k)k^2)$. The numerical data is consistent with the prediction that the time period of oscillations $T = 2\pi/(k_z U_0) = 2\pi/(10 \times 10) \approx 0.062$. In the same plot, we also exhibit the corresponding plot for $U_0 = 0$, which acts as an envelop for the $U_0 = 10$ curve. Thus we demonstrate that the renormalized viscosity $\nu(k) = 1/(\tau_c k^2)$ for $U_0 = 0$ and 10 are the same, and it is Galilean invariant, though the correlation function is a function of $\mathbf{U}_0$. Yakhot *et al.* [9] argued that the sweeping effect is negligible when the parameter $\epsilon = 0$. Note however that our proof is explicit and direct compared to previous works. Our results are very encouraging for application of Eulerian field theory to field-theoretic computations of turbulence.

References

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