

ON THE EDGE OF AN INVERSE CASCADE

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Abstract We demonstrate that systems with a parameter-controlled inverse cascade can exhibit critical behavior for which at the critical value of the control parameter the inverse cascade stops. In many dynamical systems in nature energy is transferred to small or to large length scales by a forward or inverse cascade, respectively. There are some examples, however, that have a mixed behavior such as fast rotating fluids, conducting fluids in the presence of strong magnetic fields, flows in constrained geometry, and others. In these examples the injected energy cascades both forward and inversely in fractions that depend on the value of a control parameter (rotation rate/magnetic field/aspect ratio ect). In the presented work we demonstrate using the 2D-MHD model that the transition from a forward to an inverse cascade can occur by a critical transition. In the absence of any external magnetic forcing the system reduces to hydrodynamic fluid turbulence with an inverse energy cascade. In the presence of strong magnetic forcing the system behaves as 2D-MHD turbulence with forward energy cascade. As the amplitude of the magnetic forcing is varied a critical value is met for which the energy flux towards the large scales becomes zero. Close to this point the energy flux scales as a power law with the departure from the critical point and the normalized amplitude of the fluctuations diverges. The generality of this behavior to other systems with variable inverse cascades will be discussed.

INTRODUCTION

In many dynamical systems in nature energy is transferred to smaller or to larger length scales by a mechanism known as forward or inverse cascade, respectively. In three-dimensional hydrodynamic (HD) turbulence energy cascades forward from large to small scales while in two-dimensional HD turbulence energy cascades inversely from small scales to large scales [1, 2]. There are some examples, however, that have a mixed behavior such as fast rotating fluids, stratified flows, conducting fluids in the presence of strong magnetic fields, or flows in constrained geometry [4, 5, 7, 8, 6, 9]. In these examples the injected energy cascades both forward and inversely in fractions that depend on the value of a control parameter μ (rotation rate/magnetic field/aspect ratio). In rotating flows, for example, when the rotation is weak the behavior of the flow is similar to isotropic turbulence and energy cascades forward. As the rotation rate is increased variations along the direction of rotation are suppressed and the flow starts to become quasi-2D. Eventually when rotation is strong enough the two-dimensional component of the flow dominates and energy starts to cascade inversely to the large scales. This behavior has also been observed in experimental setups [10, 11, 12, 13] but also in atmospheric boundary layers [14].

THE 2D-MHD CASE

In this work [3] we try to demonstrate the idea of an inverse cascade near criticality for a realistic model: the two-dimensional incompressible magneto-hydrodynamics (MHD) in a double periodic square domain of size $2\pi L$. The dynamical equations for the system can be written in terms of the vorticity $\omega = \mathbf{e}_z \cdot \nabla \times \mathbf{u}$ (where \mathbf{u} is the velocity field) and the vector potential a of the magnetic field $\mathbf{b} = \nabla \times (\mathbf{e}_z a)$. They are given by:

$$\begin{aligned} \partial_t \omega + \mathbf{u} \cdot \nabla \omega &= \mathbf{b} \cdot \nabla j + \nu^+ \Delta^n \omega + \nu^- \Delta^{-m} \omega + \phi_\omega \\ \partial_t a + \mathbf{u} \cdot \nabla a &= +\eta^+ \Delta^n a + \eta^- \Delta^{-m} a + \phi_a \end{aligned} \quad (1)$$

where \mathbf{e}_z is the unit vector normal to the plane, and $j = \mathbf{e}_z \cdot \nabla \times \mathbf{b}$. Energy is removed from the system by the terms proportional to ν^+ and η^+ in the small scales and by ν^- and η^- in the large scales. With these choices we are left with four control parameters. We have a Reynolds number for the forward energy cascade: $Re^+ \equiv (f_0^{1/2} k_f^{1/2-2n})/|\nu^+|$, a Reynolds number for the inverse energy cascade: $Re^- \equiv (f_0^{1/2} k_f^{1/2+2m})/|\nu^-|$, the ratio of the forcing length scale to the box size $k_f L$ and $\mu = \|\mathbf{F}_b\|/\|\mathbf{F}_u\|$ the ratio of magnetic to mechanical forcing. The last parameter controls the transition from an inverse cascade to a direct cascade. The system in the absence of forcing and dissipation conserves two positive-definite quadratic quantities: the total energy $E = \frac{1}{2} \langle \mathbf{u}^2 + \mathbf{b}^2 \rangle$ and the square vector potential $A = \frac{1}{2} \langle a^2 \rangle$. In the absence of any external magnetic field or a magnetic source ϕ_a any magnetic field fluctuations that exist at $t=0$ will die out and the system will reduce to ordinary 2D fluid turbulence with an inverse cascade for energy and a forward cascade of A that acts like the variance of a passive scalar. If, however, a magnetic force \mathbf{F}_b is sufficiently strong the flow will sustain

magnetic field fluctuations and become magnetic dominated with a forward energy cascade and an inverse cascade of A [15].

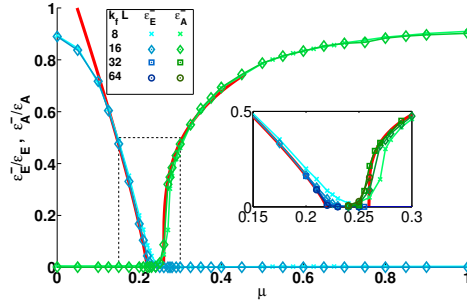


Figure 1. Normalized energy dissipation in the large scales ϵ_E^-/ϵ_E and normalized square vector potential dissipation ϵ_A^-/ϵ_A as a function of μ for different values of $k_f L$.

Figure 1 presents the large scale dissipation rates ϵ_E^- and ϵ_A^- normalized by the total injection rates $\epsilon_E = \epsilon_E^- + \epsilon_E^+$ and $\epsilon_A = \epsilon_A^- + \epsilon_A^+$ as a function of μ . For small values of μ the system behaves like HD flow with an inverse cascade of energy $\epsilon_E^-/\epsilon_E \simeq 1$. At the same time no inverse cascade of A is observed since $\epsilon_A^-/\epsilon_A \simeq 0$. For $\mu \simeq 1$, on the other hand, the system behaves like an MHD flow with no inverse cascade of energy $\epsilon_E^-/\epsilon_E \simeq 0$ but an inverse cascade of A , ($\epsilon_A^-/\epsilon_A \simeq 1$). For intermediate values of μ energy and square vector potential are dissipated both at large and small scales. Around $\mu \simeq 0.22$ the inverse cascade of energy ends and at $\mu \simeq 0.25$ the inverse cascade of A begins.

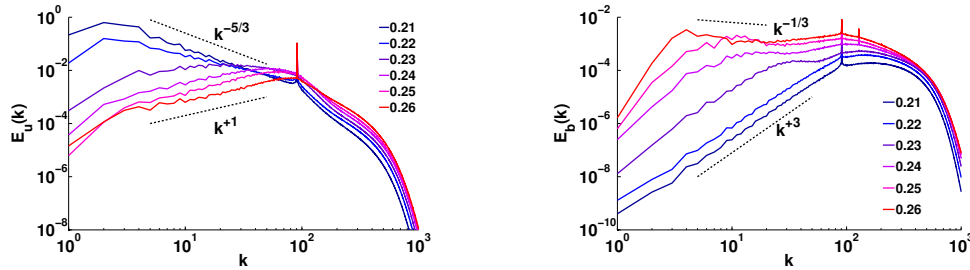


Figure 2. Kinetic energy spectra (top panel) and magnetic energy spectra (bottom panel) for $k_f L = 64$ and different values of μ varying from 0.21 to 0.26.

We note that the effect of μ on the distribution of energy in scale space. Figure 2 shows the kinetic energy spectra and the magnetic energy spectra for different values of the parameter μ varying from $\mu = 0.21$ to $\mu = 0.26$ and $k_f L = 64$. It is clear that as μ crosses the critical value μ_c the slope of the kinetic energy spectrum varies from a value close to $-5/3$ to a value that could be interpreted as $+1$ implying equipartition of kinetic energy in all modes. The slope of the magnetic energy spectrum on the other hand decreases from the positive $+3$ value to a value close to $-1/3$.

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