ON THE OPTIMAL VORTICITY FUNCTION OF VORTEX RINGS

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Abstract We present here an original approach to reconstruct the vorticity distribution inside an axisymmetric vortex ring from some incomplete and possibly noisy measurements of the corresponding velocity field. The idea is to formulate an *inverse problem* for identifying the structure of the vorticity distribution and solve this problem by a suitable numerical optimization algorithm. The novelty of the present study is that the vorticity function is reconstructed in a very general form with no assumptions other than smoothness and the behaviour at the endpoints of its domain of definition. This is fundamentally different from classical approaches (*e. g.*[13]) reducing the reconstruction problem to fitting a small number of variables parameterizing the vorticity distribution of a given vortex-ring model.

INTRODUCTION

Vortex rings are encountered in many fluid flows, ranging from biological propulsion [4] to the fuel injection in internal combustion engines [8]. In some practical applications, it is often necessary to reconstruct the vortex structure from (incomplete) measurements of the induced velocity field (obtained via experiments or direct numerical simulation). The classical approach is to fit measured data to theoretical vortex ring models. For such fitting purposes, the Norbury-Fraenkel inviscid vortex ring model [10] was largely used and proved very useful in estimating global properties of actual vortex rings [9]. We suggest in the following a new approach to reconstruct the vorticity distribution inside a vortex ring.

PHYSICAL PROBLEM AND OPTIMAL RECONSTRUCTION OF THE VORTICITY FUNCTION

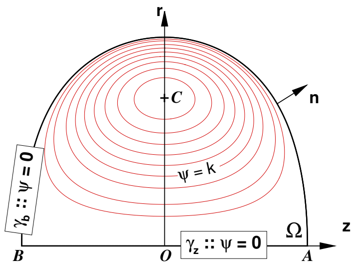

Figure 1. Domain Ω for the vortex ring problem. Contours represent the level sets of the streamfunction ψ in the frame of reference moving with the vortex ring.

Using cylindrical coordinates (z, r, θ), with z the propagation direction of the flow, a vortex ring is defined (see Figure 1) as the axisymmetric region Ω such that the azimuthal vorticity $\omega := \omega_\theta$ is $\omega \neq 0$ in Ω and $\omega = 0$ elsewhere. The domain Ω, also called *vortex bubble*, is delimited by streamline $\psi = 0$, where $\psi(r, z)$ is the Stokes streamfunction in the frame of reference moving with the vortex ring. Classical vortex ring models are stationary solutions of Euler equations. The key feature of such models is that the axisymmetric inviscid flow is completely described by (*e. g.* [2]):

$$\frac{\omega}{r} = \begin{cases} f(\psi) & \text{in} \quad \Omega, \\ 0 & \text{elsewhere,} \end{cases} \qquad \begin{cases} \mathcal{L}\psi = -r f(\psi) & \text{in} \quad \Omega, \\ \psi = 0 & \text{on} \quad \gamma := \gamma_z \cup \gamma_b \end{cases} \tag{1}$$

with $f : \mathbb{R} \to \mathbb{R}$ called *vorticity function* and $\mathcal{L}$ the self-adjoint elliptic operator $\mathcal{L} := \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)$. The only known analytical solution of this problem, the Hill's spherical vortex, assumes $f(\psi) = const$ within a spherical domain Ω.

The very popular Norbury-Fraenkel (NF) family of vortex rings was numerically constructed [10] by prescribing (see Figure 1): $f(\psi) = const$, $\forall \psi > k > 0$, and $f(\psi) = 0$, $\forall \psi \leq k$. In the following, we do not make any assumption on the vorticity function and formulate the following reconstruction problem:
Given the measurements $m : \gamma_b \cup \gamma_z \to \mathbb{R}$ of the velocity component tangential to the boundaries γ_b and γ_z, find a vorticity function $\hat{f} \in H^1(\mathcal{I})$, such that the corresponding solution of (1) *matches data m as well as possible in the least-squares sense,* i. e.

$$\hat{f} := argmin_{f \in H^1(\mathcal{I})} \mathcal{J}(f), \quad \mathcal{J}(f) := \frac{\alpha_b}{2}\int_{\gamma_b}\left(\frac{1}{r}\frac{\partial \psi}{\partial n}\bigg|_{\gamma_b} - m\right)^2 d\sigma + \frac{\alpha_z}{2}\int_{\gamma_z}\left(\frac{1}{r}\frac{\partial \psi}{\partial n}\bigg|_{\gamma_z} - m\right)^2 d\sigma, \tag{2}$$

where $\mathcal{I}$ is the "identifiability interval", $\mathcal{I} := [0, \psi_{\max}]$, with $\psi_{\max} > \max_{\mathbf{x}\in\Omega} \psi(\mathbf{x})$.
In contrast to the most common problems of this type, in which the source function depends on the independent variables (e.g., on $\mathbf{x}$), in our reconstruction formulation the source f is sought as a function of the *state variable* ψ. To address this aspect of the problem, a specialized version of the adjoint-based gradient approach was developed by deriving a convenient expression of the (Sobolev) gradient of the cost functional (2). A similar method was developed in [3] for the reconstruction of constitutive relations and already applied to other estimation problems in fluid mechanics in [11]. The corresponding numerical algorithm was implemented using the freely available finite-element software `FreeFem++` [7]. The details of the computational algorithms are available in [5].

RESULTS AND DISCUSSION

In addition to standard tests on the accuracy of the adjoint gradients, our approach was validated (see [5]) by reconstructing the vorticity functions in two classical problems, namely, the vortex rings of Hill and Norbury-Fraenkel (see figure 2 a and b). A very accurate reconstruction for the Hill vortex ring is obtained for different initial guesses of the vorticity function. Given that we required our reconstructions to be continuous (H^1) functions, the second problem involving a

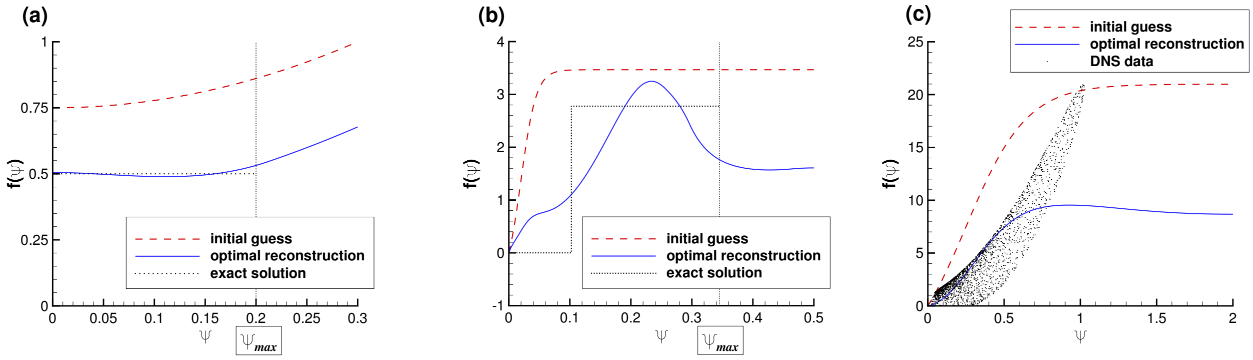

Figure 2. Reconstructed vorticity function $f(\psi)$ (blue solid lines) and the corresponding initial guesses (red dashed lines) for different test cases: (a) Hill's spherical vortex ring, (b) Norbury-Fraenkel vortex ring, (c) DNS generated vortex ring. For cases (a) and (b), the black dotted line represents exact solution, whereas the vertical dotted lines mark the maximum values achieved by the streamfunction in exact solution. For case (c), the scatter plot $\{\omega(r_p, z_p)/r_p, \psi(r_p, z_p)\}_p$ obtained from DNS data is represented by dots.

discontinuous vorticity function turned out to be particularly challenging. As can be anticipated from the analogy with the Gibbs phenomenon in numerical interpolation, the optimal reconstructions exhibit "wiggles" around the discontinuity of f. Thus, in spite of the "pathology" of this test problem, our approach behaves as expected. Since our main goal was to validate the method, the reconstruction problem was chosen to be "well-determined" in the sense that the velocity measurements m were available on the entire boundary γ of the domain (which, in particular, determined the vortex circulation Γ through Stokes' theorem). The accuracy of the reconstruction as measured by the corresponding values of the cost functional was indeed very good (figures not shown).

We also tested the reconstruction method using direct numerical simulation (DNS) data from [6]. The Reynolds number of the flow, based on the injection velocity an the diameter of the generator is 3400. Figure 2 c) shows that the reconstructed vorticity function lies inside the scatter plot $\{\omega(r_p, z_p)/r_p, \psi(r_p, z_p)\}_p$ obtained from DNS data. Experimental studies [12, 1] reported similar scatter in the plots of ω/r versus ψ, indicating that the hypothesis of the steadiness of the flow is not satisfied by real flows. However, experimental data was rather well fitted by an exponential vorticity function $f(\psi) = a\exp(b\psi)$, with a and b representing two constants adjusted during the fitting procedure [1]. This supports the idea that steady inviscid models could be used as good approximations of unsteady viscous vortex rings arising in real flows if the vorticity function $f(\psi)$ is accurately determined. Our reconstruction algorithm provides such a good approximation, proved to be better than a simple interpolation of the DNS data from the scatter plot. The applicability of the reconstruction method for high Reynolds number DNS vortex rings is currently under investigation.

References

[1] D. G. Akhmetov. *Vortex Rings*. Springer-Verlag, Berlin, Heidelberg, 2009.
[2] G. K. Batchelor. *An Introduction to Fluid Dynamics*. Cambridge University Press, Cambridge, New York, 7th edition, 1988.
[3] V. Bukshtynov, O. Volkov, and B. Protas. On optimal reconstruction of constitutive relations. *Physica D*, **240**:1228–1244, 2011.
[4] J. O. Dabiri. Optimal vortex formation as a unifying principle in biological propulsion. *Annual Review of Fluid Mechanics*, **41**(1):17–33, 2009.
[5] I. Danaila and B. Protas. Optimal reconstruction of inviscid vortices. *submitted*, **http://arxiv.org/abs/1411.7365**, 2014.
[6] I. Danaila, C. Vadean, and S. Danaila. Specified discharge velocity models for the numerical simulation of laminar vortex rings. *Theor. Comput. Fluid Dynamics*, **23**:317–332, 2009.
[7] F. Hecht. New developments in Freefem++. *Journal of Numerical Mathematics*, **20**:251–266, 2012.
[8] F. Kaplanski, S. S. Sazhin, S. Begg, Y. Fukumoto, and M. Heikal. Dynamics of vortex rings and spray-induced vortex ring-like structures. *European Journal of Mechanics B/Fluids*, **29**(3):208–216, 2010.
[9] K. Mohseni and M. Gharib. A model for universal time scale of vortex ring formation. *Phys. Fluids*, **10**:2436–2438, 1998.
[10] J. Norbury. A family of steady vortex rings. *J. Fluid Mech.*, **57**:417–431, 1973.
[11] B. Protas, B. R. Noack, and M. Morzyński. An optimal model identification for oscillatory dynamics with a stable limit cycle. *J. Nonlin. Sci.*, **24**:245–275, 2014.
[12] J. P. Sullivan. Study of a vortex ring using a laser doppler velocimeter. *AIAA Journal*, **11**:1384, 1973.
[13] Y. Zhang and I. Danaila. A finite element bfgs algorithm for the reconstruction of the flow field generated by vortex rings. *Journal of Numerical Mathematics*, **20**:325–340, 2012.