NON-GAUSSIANITY IN TURBULENT PAIR DISPERSION

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Abstract We consider an extension of Thomson's [8] two-particle Lagrangian stochastic model that is constructed to be consistent with the 4/5 law of turbulence. It is shown that one effect of non-zero skewness in the longitudinal relative velocity is to reduce the value of Richardson's constant by approximately a factor of two relative to the model with zero skewness and that this value is close to recent measurements from direct numerical simulation of homogeneous isotropic turbulence.

INTRODUCTION

The dispersion of pairs of particles in a turbulent fluid is a celebrated problem in the study of turbulence. Richardson [5] studied the problem experimentally and suggested that relative dispersion is governed by a diffusivity which is proportional to $r^{4/3}$ where r is the absolute separation of a pair. The solution of the associated diffusion equation is a probability density function (pdf) for r which is non-Gaussian but self-similar. This leads to the celebrated result that $\langle r^2 \rangle \propto t^3$ once the initial separation, r_0, is forgotten. (This result can also be derived using dimensional arguments asssuming Kolmogorov's hypotheses for homogeneous isotropic turbulence.) As the veracity of this result in real turbulence is still unknown, relative dispersion remains an active field of research not only for its theoretical interest but also because of its practical importance via the relationship between relative dispersion and concentration fluctuations.

LAGRANGIAN STOCHASTIC MODEL

Consider two particles with respectively position $\boldsymbol{x}_i$ and velocity $\boldsymbol{u}_i$ $(i = 1, 2)$ at time t. The trajectory of the pair can be represented by the six-dimensional vector $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2)$ and its velocity by $\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2)$. The evolution of $(\boldsymbol{x}, \boldsymbol{u})$ is given by

$$\begin{aligned} \mathrm{d}u_i &= a_i(\boldsymbol{u}, \boldsymbol{x}, t)\,\mathrm{d}t + b_{ij}\,\mathrm{d}W_j(t), \quad i = 1, \ldots, 6 \\ \mathrm{d}\boldsymbol{x} &= \boldsymbol{u}\,\mathrm{d}t \end{aligned} \tag{1}$$

where $\boldsymbol{W}$ is a vector-valued Wiener process, b_{ij} is chosen to be $\sqrt{C_0 \varepsilon}\,\delta_{ij}$ for consistency with the Lagrangian velocity structure function in the inertial subrange, C_0 is the constant of proportionality in this structure function and ε is the mean kinetic energy dissipation rate. The well-mixed condition [7] constrains the model to be consistent with the Eulerian velocity statistics and leads to an appropriate form of the drift term.
In order to determine $\boldsymbol{a}(\boldsymbol{u}, \boldsymbol{x}, t)$ we use the Fokker-Planck equation corresponding to (1) and, following [7, 8], partition $\boldsymbol{a}$ into $\boldsymbol{a}^{(1)}$, which satisfies a balance between an advective term in velocity-space and the diffusive term of the Fokker-Planck equation, and $\boldsymbol{a}^{(2)}$ which satisfies the non-diffusive part of the Fokker-Planck equation (which is the exact transport equation satisfied by the Eulerian velocity pdf, $p_E(\boldsymbol{u})$).

THE EULERIAN VELOCITY PDF

We note that in 3-D the model reduces to a quasi-two-dimensional model in isotropic turbulence: it is thus simpler to work in spherical polar coordinates. We assume that $p_E(\boldsymbol{u})$ is separable so that for isotropic turbulence

$$p_E(u_\parallel, u_\perp, u_\vdash) = \frac{p_{u_p}(u_p)\,p_{u_\parallel}(u_\parallel)}{2\pi u_p} = p_{u_\perp}(u_\perp)\,p_{u_\vdash}(u_\vdash) p_{u_\parallel}(u_\parallel) \tag{2}$$

where $u_\parallel$ is the longitudinal component of $\boldsymbol{u}$ and $u_p = \sqrt{u_\perp^2 + u_\vdash^2}$, where $u_\perp$ and $u_\vdash$ are the perpendicular components. It follows from the circular symmetry of $u_\perp$ and $u_\vdash$ that $p_{u_\perp}(u_\perp) \sim N(0, \sigma_\perp^2)$ where $\sigma_\perp^2 = (4/3)C(\varepsilon r)^{2/3}$ and $C = 2$ is the Kolmogorov constant. While evidence from direct numerical simulation (DNS) of relative dispersion indicates that $p_E(\boldsymbol{u})$ is not separable [4, 6] it is a necessary step to make the model tractable. It will be of interest to see how well the model performs despite this limitation.
The longitudinal pdf, $p_{u_\parallel}$, is derived following an approach used in modelling single-particle dispersion in the convective atmospheric boundary layer. In this case the pdf is positively skewed i.e. there is a relatively low probability of strong updraughts versus a high probability of weak downdraughts. Here, $p_{u_\parallel}$ is negatively skewed. Following [1, 3, 2], we specify $p_{u_\parallel}$ as the weighted sum of two Gaussian distributions:

$$p_{u_\parallel} = \sum_{i=1}^{2} \frac{A_i}{\sqrt{2\pi}\sigma_{u_\parallel i}} \exp\left(-\frac{(u_\parallel - \overline{u}_{\parallel i})^2}{2\sigma_{u_\parallel i}^2} \right) \tag{3}$$

where A_i, $\sigma_{u_\parallel i}$ and $\overline{u}_{\parallel i}$ are as yet unspecified. In order to determine these unknowns and to construct a pdf that gives the correct first four moments, $p_{u_\parallel}$ must satisfy

$$\int_{-\infty}^{\infty} p_E(u_\parallel, u_\perp, u_\vdash)\,\mathrm{d}^3\boldsymbol{u} = 1, \qquad \int_{-\infty}^{\infty} u_\parallel p_E(u_\parallel, u_\perp, u_\vdash)\,\mathrm{d}^3\boldsymbol{u} = 0, \tag{4}$$

$$\int_{-\infty}^{\infty} u_\parallel^2 p_E(u_\parallel, u_\perp, u_\vdash)\,\mathrm{d}^3\boldsymbol{u} = C(\varepsilon r)^{2/3}, \qquad \int_{-\infty}^{\infty} u_\parallel^3 p_E(u_\parallel, u_\perp, u_\vdash)\,\mathrm{d}^3\boldsymbol{u} = \frac{4}{5}\varepsilon r. \tag{5}$$

It is also assumed that $\overline{u}_{\parallel i} = \alpha_i \sigma_{u_\parallel i}$ where $\alpha_1 = -\alpha_2 = 1$. Analytical forms of A_i and $\sigma_{u_\parallel i}$ are found by substituting (2) and (3) into (4)–(5) and solving for A_i and $\sigma_{u_\parallel i}$. Once $p_{u_\parallel}$ is known, $\boldsymbol{a}^{(2)}$ can be obtained from the transport equation for p_E.

RESULTS

DNS results suggest $5 \lesssim C_0 \lesssim 7$; here we choose $C_0 = 6$. Model statistics are computed with 100,000 pairs using an adaptive time step. As expected, once r_0 is forgotten, the mean-square separation $\langle r^2 \rangle$ grows like $g\varepsilon t^3$ where g is a constant. The value of g is much sought after: both DNS and experimental values are the subject of considerable uncertainty due largely to the lack of a sufficiently long inertial subrange. To date $g \approx 0.5$ is often taken to be the best available estimate. Comparing the Gaussian and non-Gaussian versions of the model in figure 1a, we see that the effect of non-zero skewness in the Eulerian velocity distribution is to reduce the value of g by approximately a factor of two.

The skewness of r is shown in figure 1b. Since r can never decrease below zero, in both cases the pdf of r is positively skewed. In the non-Gaussian case there is an increased probability of particles moving towards each other with large velocities (compared with the Gaussian case). Hence, the skewness is reduced in the non-Gaussian case. A similar difference can be seen in the kurtosis of r.

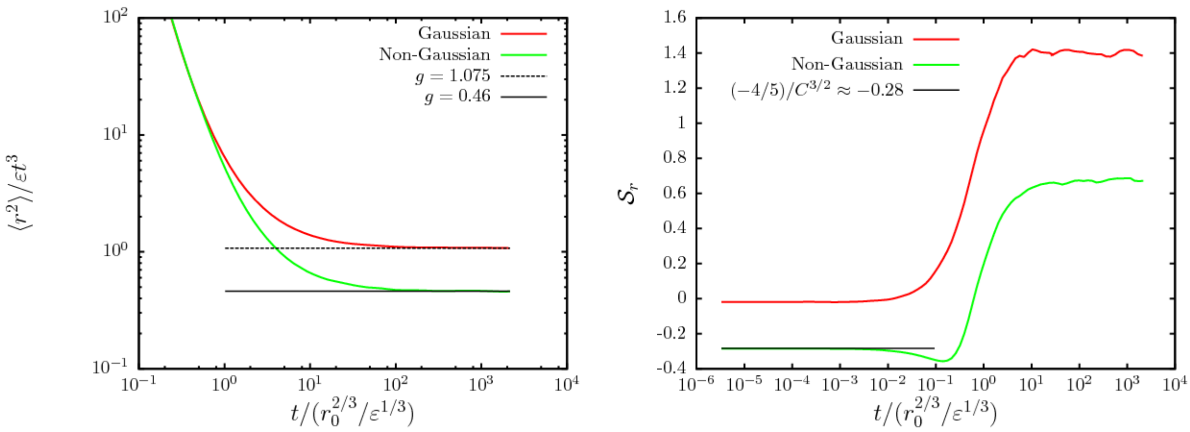

Figure 1. For both Gaussian and non-Gaussian versions of the model: (a) compensated plot of $\langle r^2 \rangle$; (b) skewness of r, S_r.

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