A STATISTICAL MECHANICS FRAMEWORK FOR THE LARGE-SCALE STRUCTURE OF A TURBULENT VON KÁRMÁN FLOW

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Abstract Recent experimental results on large scale coherent steady states observed in experimental turbulent von Kármán flows are revisited from a statistical mechanics perspective. We argue that the coherent steady states may be described as the equilibrium states of ad hoc lattice models, that can be used to define global properties of von Kármán flows, such as their temperatures, their fugacity and so on. The equilibrium description is then enlarged, in order to reinterpret a series of results about the stability of those steady states, their susceptibility to symmetry breaking, in the light of a deep analogy with the Curie-Weiss statistical theory of Ferromagnetism.

Describing the complexity of turbulent flows with tools borrowed from statistical mechanics is a long-standing dream of theoreticians. In 1949, five years after the publication of its solution for the problem of phase transition in the 2D Ising model, Onsager published a notorious study of the statistical mechanics of the point vortex model [8], a special solution of the 2D Euler equations that allows to interpret the emergence of long lived coherent structures in terms of the pairing between vortices mutually interacting through a long range Coulombian potential. Since Onsager, the description of 2D turbulence using statistical mechanics has greatly improved. Starting from the seminal work of Robert-Miller-Sommeria [5, 9] and subsequent work from then [4, 2], it led to a description of the coherent structures that seems to match the observed large scale organization in experimental and numerical 2D turbulence. However, the 3D case still escapes theoretical grasp.
A first light of hope may come from the special case of von Kármán (VK) turbulence, a now classical small size experiment that reaches Reynolds numbers of the order of 10^6 through the stirring of a fluid in between two counter-rotating propellers. At such Reynolds number, turbulence is traditionally expected to be fully developed with a wide range of interacting scales. Indeed, previous analysis of turbulence properties in the middle shear layer evidenced scaling properties and intermittency corrections in agreement with other measurements in fully developed turbulent flows in different geometries [1]. Some indications exist though, that the number of effective degrees of freedom in a turbulent VK flow is not so large : at Reynolds number around $10^5 - 10^6$, one can perform Poincaré maps of the torque exerted by the turbulence on each propeller and exhibit attractors, limit cycles[10], that are traditionnally usually observed in dynamical systems, with only a few degrees of freedom. This suggests that only a few global quantities would be able to describe the system so that some kind of statistical mechanics approach should be used in this system, in order to identify hydrodynamical analogues for "temperatures" or "chemical potentials".

"Ferro-Magnetic" Quantity	Hydrodynamic Analog
Spin	"Beltrami Spin"
Magnetization	Angular momentum and circulation
Thermostat	Forcing and dissipation
Temperature	Velocity fluctuations
Symmetry breaking fields	Rotation and Torque numbers
Susceptibility	Response to asymmetry

Table 1. Analogy between a two components spin system and the VK flow

In this work, we argue that the steady topologies observed in a turbulent VK set-up, as well as the transitions between those, could be interpreted in terms of a statistical physics modeling. This allows us to exhibit a deep analogy between the VK large scale dynamics and the standard behavior of Ferro-Magnetic material (see table), with diverging susceptibility and spontaneous symmetry breaking at a critical temperature (see Fig. 1). We show that the topologies of the VK steady states could be interpreted thermodynamically, using as a guideline an equilibrium theory that relies on the axially symmetric Euler equations and an adequate modeling of the fluctuations. At first sight, there is no reason to expect that such an equilibrium theory could be relevant for the VK cases : the lack of axial-symmetry for the instantaneous VK dynamics, the presence of a large-scale forcing which precludes any kind of separation of scale working hypothesis, the non-gaussianity of the VK fluctuations and the UV catastrophe associated to statistical theories based on the Euler axially symmetric equations are for example as many reasons which should imply that such an approach is bound to fail.
However, in the present case, the forcing and dissipation that are present in the experiment seem to play the role of

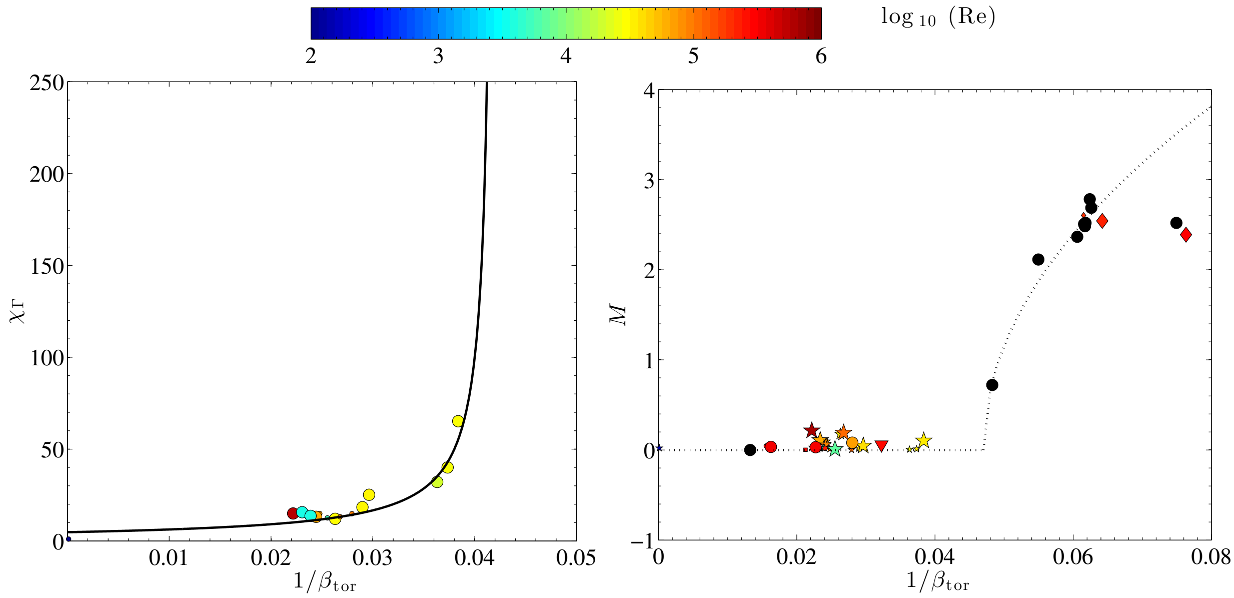

Figure 1. Left: The VK susceptibility as a function of the temperature. The black dotted line is a fit $b/(1/\beta - 1/\beta^{\star})$, with $b = 0.2$ and $1/\beta_{\star} = 0.044{\pm}0.03$; Right : The VK magnetization as a function of the toroidal temperature for different propellers (TM87(+)(circles) and (-) (diamond) and TM73 (+)(triangles) and (-) (stars)) . The black dotted line indicates a fit $y = a\sqrt{1/\beta - 1/\beta^{\star}}$, with $1/\beta_{\star} = 0.044 \pm 0.03$, $a = 21$. The black circles are magnetization estimates based on the height of the hysteresis cycle . In both figures, the color codes the Reynolds number.

thermostats, which prescribe strong correlations between the degrees of freedom present in the flow. This allows for a statistical mean-field interpretation of the mean flow, provided one plugs into the theory additional ansatz non prescribed by the Euler equations. In other words, the VK steady states do *not* correspond to ideal axially symmetric equilibria. Still, to first order, there exists some good indication that those can be interpreted as maximizers over an appropriate restricted subset of phase space of a suitably defined configuration entropy. This experimental fact provides an intuitive ground, to think about the VK experiment in terms of a statistical mean field theory [6, 7, 11].. We show here that this analogy goes beyond the mere description of the VK topologies and that the transitions between the various topologies could be phrased in terms of phase transition and spontaneous symmetry breaking [3] shedding an unexpected analogy between VK turbulence and the theory of ferro-magnetism (for details see *Thalabard et al., arXiv:1501.01182 [physics.flu-dyn]*).

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