EXACT TWO-DIMENSIONALIZATION OF LOW-MAGNETIC-REYNOLDS-NUMBER FLOWS SUBJECT TO A STRONG MAGNETIC FIELD

Basile Gallet[1], Charles R. Doering [2]
[1]*Laboratoire SPHYNX, Service de Physique de l'État Condensé, DSM, CEA Saclay, CNRS, 91191 Gif-sur-Yvette, France*
[2]*Department of Physics, Department of Mathematics, and Center for the Study of Complex Systems, University of Michigan, Ann Arbor, MI 48109, USA*

Abstract We investigate the behavior of flows, including turbulent flows, driven by a horizontal body-force and subject to a vertical magnetic field, with the following question in mind: for very strong applied magnetic field, is the flow mostly two-dimensional, with remaining weak three-dimensional fluctuations, or does it become *exactly* 2D, with no dependence along the vertical?

We restrict attention to low-magnetic-Reynolds number (Rm) flow. Because liquid metals have low magnetic Prandtl number, such low-Rm flows can have a kinetic Reynolds number as large as $Re \simeq 10^6$ and therefore be strongly turbulent.

We first focus on the quasi-static approximation, i.e. the asymptotic limit of vanishing magnetic Reynolds number $Rm \to 0$: we prove that the flow becomes exactly 2D asymptotically in time, regardless of the initial condition and provided the interaction parameter N is larger than a threshold value. We call this property *absolute two-dimensionalization*: the attractor of the system is necessarily a (possibly turbulent) 2D flow.

We then consider the full-magnetohydrodynamic equations and we prove that, for low enough Rm and large enough N, the flow becomes exactly two-dimensional in the long-time limit provided the initial vertically-dependent perturbations are infinitesimal. We call this phenomenon *linear two-dimensionalization*: the (possibly turbulent) 2D flow is an attractor of the dynamics, but it is not necessarily the only attractor of the system. Some 3D attractors may also exist and be attained for strong enough initial 3D perturbations.

These results shed some light on the existence of a dissipative anomaly for magnetohydrodynamic flows subject to a strong external magnetic field.

Liquid metal turbulence is encountered in various situations, ranging from metallurgy [1] to the flow in the Earth's outer core [2], including laboratory experiments on magnetohydrodynamic (MHD) turbulence [3, 4]. Because their kinematic viscosity is much lower than their magnetic diffusivity, with magnetic Prandtl numbers Pm typically in the range of 10^{-6} to 10^{-5}, flows of liquid metals can be turbulent and still be characterized by a small magnetic Reynolds number: in most laboratory experiments on MHD turbulence, the kinetic Reynolds number is in the range $10^4 - 10^6$, with a magnetic Reynolds number rarely exceeding unity. At small enough scales, turbulence in Earth's outer core may also be considered as a low-Rm flow.

BODY-FORCED MAGNETOHYDRODYNAMICS

We consider the setup sketched in the left-hand panel of figure 1: an incompressible fluid of kinematic viscosity ν, density ρ and electrical conductivity σ flows inside a domain $(x, y, z) \in \mathcal{D} = [0, L] \times [0, L] \times [0, H]$ with a Cartesian frame $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$. It is stirred by a time-independent divergence-free two-dimensional horizontal body-force $\mathbf{f}(x, y) = (f_x, f_y, 0)$ that is periodic on a scale ℓ, an integer fraction of L. That is, $\mathbf{f}(x, y) = \boldsymbol{\phi}(\frac{x}{\ell}, \frac{y}{\ell})$, where $\boldsymbol{\phi}$ is periodic of period 1 in each dimensionless variable.

A homogeneous steady vertical magnetic field $B_0\mathbf{e}_z$ is applied to the system. The magnetic field inside the fluid follows the induction equation, which using the decomposition $\mathbf{B}(x, y, z, t) = B_0\mathbf{e}_z + \mathbf{b}(x, y, z, t)$ reads

$$\partial_t \mathbf{b} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{b} - (\mathbf{b} \cdot \boldsymbol{\nabla})\mathbf{u} = B_0 \partial_z \mathbf{u} + \eta \Delta \mathbf{b} \,, \qquad \boldsymbol{\nabla}.\mathbf{b} = 0 \,, \tag{1}$$

where $\eta = 1/(\mu_0 \sigma)$, with μ_0 the magnetic permeability of vacuum. The velocity field follows the body-forced Navier-Stokes equation with the Lorentz force

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \nu \Delta \mathbf{u} + \mathbf{f}(x, y) + \frac{1}{\rho \mu_0} (\boldsymbol{\nabla} \times \mathbf{b}) \times (B_0 \mathbf{e}_z + \mathbf{b}) \,, \qquad \boldsymbol{\nabla}.\mathbf{u} = 0 \,. \tag{2}$$

From equations (1) and (2) we define the kinetic and magnetic Reynolds numbers, and the interaction parameter N, using the root-mean-square velocity U, where the mean is performed over space and time:

$$Re = \frac{U\ell}{\nu} \,, \qquad Rm = \frac{U\ell}{\eta} \,, \qquad N = \frac{\sigma B_0^2 \ell}{\rho U} \,. \tag{3}$$

N is a dimensionless measure of the strength of the external field B_0.

A simpler set of equations can be obtained in the quasi-static limit $Rm \to 0$,

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \nu \Delta \mathbf{u} + \mathbf{f}(x, y) - \beta \Delta^{-1} \partial_{zz} \mathbf{u} \,, \qquad \boldsymbol{\nabla}.\mathbf{u} = 0 \,, \tag{4}$$

where β is the inverse Joule time scale corresponding to magnetic damping, $\beta = \sigma B_0^2/\rho$. It is related to the interaction parameter through $N = \beta\ell/U$. The main convenience of the quasi-static equation (4) is that it involves the velocity field only.
The magnetohydrodynamic problem (1-2) and its quasi-static version (4) admit two-dimensional solutions of the form $\mathbf{u} = \mathbf{V}(x, y, t)$, $\mathbf{b} = 0$, where $\mathbf{V}(x, y, t)$ satisfies a standard non-magnetic two-dimensional Navier-Stokes equation. We prove that, for large enough interaction parameter, such motion is robust with respect to three-dimensional perturbations, i.e., the three-dimensional part of the velocity field decays in the long-time limit, and the system relaxes to a 2D flow.

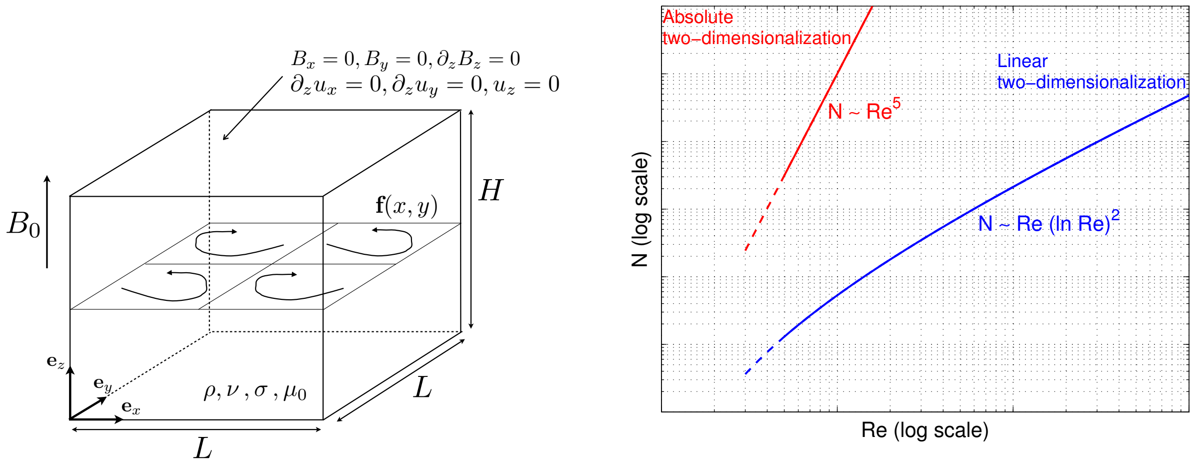

Figure 1. Left-hand panel: flow of an electrically-conducting newtonian fluid subject to a uniform vertical magnetic field B_0. It is driven by a horizontal body-force $\mathbf{f}$ that is independent of the vertical. we assume periodic boundary-conditions in the horizontal, and stress-free boundaries with large magnetic-permeability at $z = 0$ and $z = H$. **Right-hand panel:** a sketch of the bounds in parameter space, for the quasi-static case and single-mode forcing. Above the linear two-dimensionalization line, the flow has a stable 2D attractor. However, this attractor is stable for small 3D perturbations only, and fully 3D attractors of the system may exist as well. Above the absolute two-dimensionalization line, the flow has only 2D attractors.

RESULTS

Using rigorous analysis and estimates, we derive sufficient criteria on N and Rm for the 2D flow to be robust to 3D perturbations. An example of the results obtained in the quasi-static approximation is provided in the right-hand panel of figure 1, for a body force $\mathbf{f}$ corresponding to a single horizontal Fourier-mode. We show the regions of the parameter space (Re, N) in which we prove two-dimensionalization: above the lower curve the 2D flow is stable with respect to infinitesimal 3D perturbations, i.e., the possibly turbulent 2D flow is an attractor of the dynamics. Above the upper curve, the system is robust to 3D perturbations of arbitrary amplitude, and the flow is necessarily 2D.
The asymptotic state of turbulence in the limit of infinitely strong applied B_0 and vanishing viscosity therefore depends on how one takes the double limit $(Re, N) \to (\infty, \infty)$. We show that if one takes this limit with $N \gtrsim Re(\ln Re)^2$, the flow has a 2D attractor with an energy dissipation rate per unit mass $\epsilon \lesssim \nu\frac{U^2}{\ell^2}$, i.e., there is no energy dissipative anomaly in this limit of MHD turbulence.
More general results on the full MHD equations will also be presented [5].

References

[1] P.A. Davidson. *Magnetohydrodynamic in materials processing*. Annu. Rev. Fluid Mech., **31**, 273-300, (1999).
[2] H.K. Moffatt. *Magnetic Field Generation in Electrically Conducting Fluids*. Cambridge University Press, (1978).
[3] A. Alemany, R. Moreau, P.L. Sulem and U. Frisch. *Influence of an external magnetic field on homogeneous MHD turbulence*. J. Mécanique, **18**, 2, (1979).
[4] B. Gallet, M. Berhanu and N. Mordant. *Influence of an external magnetic field on forced turbulence in a swirling flow of liquid metal*. Phys. Fluids, **21**, 085107, (2009).
[5] B. Gallet and C.R. Doering. *Exact two-dimensionalization of low-magnetic-Reynolds-number flows subject to a strong magnetic field*. submitted.