TURBULENT SUPER-DIFFUSION AS A BALLISTIC CASCADE

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Abstract Since the pioneering work of Richardson in 1926, later refined by Batchelor and Obukhov in 1950, it is predicted that the rate of separation of pairs of fluid elements in turbulent flows with initial separation at inertial scales, grows ballistically first (Batchelor regime), before undergoing a transition towards a super-diffusive regime where the mean-square separation grows as t^3 (Richardson regime). Richardson empirically interpreted this super-diffusive regime in terms of a non-Fickian process with a scale dependent diffusion coefficient (the celebrated Richardson's "4/3rd" law). However, the actual physical mechanism at the origin of such a scale dependent diffusion coefficient remains unclear. The present work proposes a simple physical phenomenology for the Richardson super-diffusivity in turbulence based on a scale dependent *ballistic* scenario rather than a scale dependent *diffusive* scenario. It is shown that this phenomenology elucidates several aspects of turbulent dispersion: (i) it gives a simple physical explanation of the origin of the super diffusive t^3 Richardson regime as an iterative cascade of scale-dependent ballistic separations, (ii) it simply relates the Richardson constant to the Kolmogorov constant (and eventually to a ballistic persistence parameter), (iii) it gives a simple physical interpretation of the non-Fickian scale-dependent diffusivity coefficient as originally proposed by Richardson and (iv) a further extension of the phenomenology, taking into account higher order corrections to the local ballisitic motion, gives a robust interpretation of the assymetry between forward and backward dispersion, with an explicit connection to the energy flux accross scales.

INTRODUCTION

In his seminal article on relative dispersion in 1926 [1], Richardson gave an interpretation of turbulent super-diffusion in terms of a non-Fickian process which could be locally modeled as a normal diffusion process, but with a scale dependent diffusion coefficient K which depends on particle separation D, according to the celebrated Richardson's 4/3rd law : $K(D) \propto D^{4/3}$. Besides, Richardson showed that this non-Fickian diffusion resulted in a cubic super-diffusive growth of the mean square separation of pairs of particles according to the law $\left\langle D^2 \right\rangle = g\epsilon t^3$, where ϵ is the turbulent energy dissipation rate and g a universal constant since known as the Richardson constant. In the framework of K41 phenomenology of turbulence the t^3 dependency can be understood as a simple dimensional constraint, when initial separation is ignored. Richardson's work was later refined by Batchelor and Obukhov in the 1950s [2], who pointed that while the loss of memory of initial separation is a reasonable assumption for the long-term dispersion, initial separation must play a role in the short-term. They showed that the rate of separation of pairs of fluid elements in turbulent flows with initial separation $\vec{D}_0$ at inertial scales must obey the following scalings :

$$R^2 = \left\langle \left(\vec{D} - \vec{D}_0\right)^2 \right\rangle = \begin{cases} S_2(\vec{D}_0)t^2 & \text{if } t < t_0 \quad (1a) \\ g\epsilon t^3. & \text{if } t > t_0 \quad (1b) \end{cases}$$

with $S_2(\vec{r}) = \left\langle |\delta_{\vec{r}}\vec{u}|^2 \right\rangle$ the full second order Eulerian structure function of the velocity field (with $\delta_{\vec{r}}\vec{u}$ the increment between two points separated by a vector $\vec{r}$ of the eulerian velocity field of the flow ; note that homogeneity is assumed, so that velocity increment only depends on the separation vector) and t_0 a characteristic time scale of the particles motion at the initial scale D_0. In K41 framework, inertial scalings for S_2 and t_0 are $S_2(\vec{D}_0) \propto \epsilon^{2/3} D_0^{2/3}$ (where local isotropy is also assumed so that $S_2(\vec{r})$ only depends on the norm of the separation vector $r = \sqrt{|\vec{r}|^2)}$ and $t_0 \propto \epsilon^{-1/3} D_0^{2/3}$ (t_0 then represents the eddy turnover time at scale D_0). Formally speaking, the initial ballistic regime (eq. 1a) is nothing but the leading term of the Taylor expansion for the mean square pair separation at short times, expressed in terms of the initial mean square relative velocity between particles. Note that such a ballistic Taylor expansion is a general and purely kinematic relation valid for any early dispersion process and is not limited to the case of turbulence. Specificities of turbulence only appear when expliciting the form of the structure function S_2 at inertial scales. This short-term ballistic regime has been shown to be accurately and robustly followed in experiments of relative pair dispersion within the inertial scales of 3D-turbulence [3]. For times exceeding t_0, a transition is expected towards an enhanced dispersion regime, cubic in time and independent of initial separaion, as originally predicted by Richardson. The Richardson constant g in eq. 1b is one of the most fundamental constants in turbulence (together with the Kolmogorov constant C_2). It plays a major role in turbulent dispersion and mixing processes. Most recent high resolution direct numerical simulations seem to point toward a robust estimate of $g \sim 0.5 - 0.6$ [4, 5, 6], in agreement with the experiment by Ott & Mann [7].

A BALLISTIC CASCADE PHENOMENOLOGY

I propose here a very simple phenomenology for Richardson's super-diffusivity, built on important previous works emphasizing the possible leading role of short term ballisitic processes [8, 9, 3, 10]. The main idea behind the dispersive process proposed here is that of an iterative ballistic scenario, as illustrated in figure 1a: a set of particle pairs with a given initial separation D_0 starts to disperse ballistically, with a separation rate $S_2(D_0)$ over a given period t_0 after which the

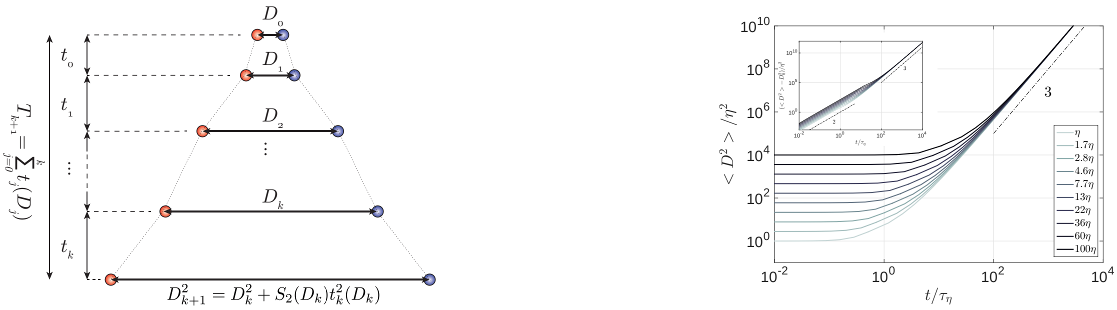

Figure 1. (a) Iterative ballisitic scheme. (b) Mean square separation as a function of time predicted by the iterative ballisitic phenomenology, for different initial separations. In particular, the transition towards the Richardson cubic separation is well captured.

mean square separation has grown to $D_1^2 = D_0^2 + S_2(D_0)t_0^2$ (following the elementary short term ballisitic regime, as given by eq 1a), then instead of considering for $t > t_0$ a sudden transition towards an enhanced cubic dispersion regime (as in eq. 1b), the same elementary ballistic process is iterated, but starting from the new initial mean square separation D_1^2, hence with a new separation rate $S_2(\vec{D}_1)$ which operates over a new period of time t_1 and so on. Thus, in this scenario the time evolution of particles mean square separation is simply described by the iterative scheme :

$$D_{k+1}^2 = D_k^2 + S_2(D_k)t'_k(D_k) \quad \text{with} \quad \begin{cases} S_2(D_k) = C\epsilon^{2/3}D_k^{2/3} \\ t'_k(D_k) = \alpha t_k = \alpha S_2(D_k)/2\epsilon \end{cases}, \tag{2}$$

where $D_k^2 =< |\vec{D}_k|^2 >$ represents the mean square separation of pairs at the k^{th} iteration step, $t'_k(D_k)$ is a scale dependent "time of flight" characteristic of the duration of the ballistic motion at step $k+1$. $S2(D_k)$ and $t'_k(D_k)$, are given by K41 scalings, with $alpha$ a parameter characteristic of the persistence of the local ballistic separation. Substituting the explicit expressions for $S_2(D_k)$ and $t'_k(D_k)$ in (2) into the iteration equation for D_k^2 leads to a simple geometrical progression (and hence to an exponential growth with the iteration number) both for the mean square separation D_k^2 and the ballisitic time scale t_k, which result in the overall separation law :

$$D_k^2 = g\epsilon\left[T_k + \left(\frac{D_0^2}{g\epsilon}\right)^{1/3}\right]^3 \quad \text{with} \quad g = \left[2\frac{(1+\frac{\alpha^2C^3}{\cdot}4)^{1/3}-1}{\alpha C}\right]^3 \tag{3}$$

where $T_k = \Sigma_j^{k-1} t'_k$ represents the total time up to the k^{th} iteration.

DISCUSSION

Interestingly, eq. 3 shows that this very simple iterative ballisitic phenomenology trivially builds a t^3 long term dispersive regime $D_k^2 = gT_k^3$, where the Richardson constant is directly related to the Kolmogorov constant C and the persistence parameter α. During ETC15, I will present a quantitative comparison of this simple model predictions with existing numerical and experimental data, which validates the proposed scenario as a realistic description of turbulent superdiffusion. I will also show how this approach can be related to the original non-Fickian idea by Richardson. Besides, I will present a simple extension of the phenomenology accounting for the wellknown backward / forward temporal asymetry of pair dispersion in turbulence. Alltogether, the present phenomenology therefore builds a simple connection between the Lagrangian problem of pair dispersion and the usual Eulerian approach of turbulent energy cascade, where the mean square separation rate is directly related to the Eulerian energy spectrum (or equivalently to S_2) and the Kolmogorov constant, while the temporal asymetry is related to the energy flux accross scales (hence with differents trends for instance for a 3D direct cascade and a 2D inverse cascade).

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