PRANDTL NUMBER DEPENDENCE OF KINETIC-TO-MAGNETIC DISSIPATION RATIO

Axel Brandenburg[1,2]
[1]*Nordita, KTH Royal Institute of Technology and Stockholm University, 10691 Stockholm, Sweden*
[2]*Department of Astronomy, AlbaNova University Center, Stockholm University, 10691 Stockholm, Sweden*

Abstract Using direct numerical simulations of three-dimensional hydromagnetic turbulence, either with helical or non-helical forcing, we show that the ratio of kinetic-to-magnetic energy dissipation always increases with the magnetic Prandtl number, i.e., the ratio of kinematic viscosity to magnetic diffusivity. This dependence can always be approximated by a power law, but the exponent is not the same in all cases. For non-helical turbulence, the exponent is around 1/3, while for helical turbulence it is between 0.6 and 2/3. In the statistically steady state, the rate of the energy conversion from kinetic into magnetic by the dynamo must be equal to the Joule dissipation rate. We emphasize that for both small-scale and large-scale dynamos, the efficiency of energy conversion depends sensitively on the magnetic Prandtl number, and thus on the microphysical dissipation process. To understand this behavior, we also study shell models of turbulence and one-dimensional passive and active scalar models. We conclude that the magnetic Prandtl number dependence is qualitatively best reproduced in the one-dimensional model as a result of dissipation via localized Alfvén kinks.

TURBULENT ENERGY DISSIPATION

One of the central paradigms of hydrodynamic turbulence is the equivalence of large-scale energy injection and small-scale dissipation into heat through viscosity—regardless of how small its value. However, magnetic fields provide an additional important pathway for dissipating turbulent energy through Joule heating. The heating rates for both viscous and Joule dissipation are proportional to the microphysical values of viscosity ν and magnetic diffusivity η, respectively. The ratio of these coefficients is the magnetic Prandtl number, $\mathrm{Pr}_M = \nu/\eta$. As these coefficients are decreasing, the velocity and magnetic field gradients sharpen just enough so that the heating rates remain independent of these coefficients.

While this picture is appealing and seemingly well confirmed, at least in special cases such as for fixed values of Pr_M, questions have arisen in those cases when the magnetic and fluid Reynolds numbers are changed in such a way that their ratio changes. Hydromagnetic turbulence simulations exhibiting dynamo action have shown that the values of energy dissipation are then no longer constant, and that their ratio scales with Pr_M [1, 2, 3, 4]. A sketch showing the transfers in and out of the two energy reservoirs, $E_\mathrm{K} = \langle \rho \boldsymbol{u}^2/2 \rangle$ and $E_\mathrm{M} = \langle \boldsymbol{B}^2/2\mu_0 \rangle$, is given in Figure 1. From this it is clear that, in the steady state, the quantity $-\langle \boldsymbol{u} \cdot (\boldsymbol{J} \times \boldsymbol{B}) \rangle$ must be positive and equal to $\langle \eta\mu_0 \boldsymbol{J}^2 \rangle$.

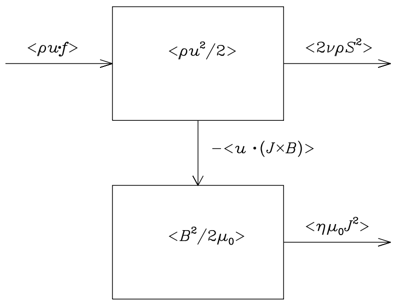

Figure 1. Sketch showing the flow of energy injected by the forcing $\langle \rho \boldsymbol{u} \cdot \boldsymbol{f} \rangle$ and eventually dissipated viscously and resistively via the terms ϵ_K and ϵ_M. Note that in the steady state, ϵ_M must be balanced by $-\langle \boldsymbol{u} \cdot (\boldsymbol{J} \times \boldsymbol{B}) \rangle$.

As in [3], ϵ_K and ϵ_M are normalized by their sum, $\epsilon_T = \epsilon_K + \epsilon_M$, which in turn is expressed in terms of the non-dimensional quantity $C_\epsilon = a\epsilon_T / \langle \rho u_\mathrm{rms}^3 k_\mathrm{f} \rangle$, where $a = 9\pi\sqrt{3}/4 \approx 12.2$ is a coefficient. First of all, note that in all cases the energy ratio $E_\mathrm{K}/E_\mathrm{M}$ is roughly independent of Pr_M but it varies with Re_M, as was demonstrated previously for the small-scale dynamo [5]. For large-scale dynamos, the ratio $E_\mathrm{K}/E_\mathrm{M}$ is essentially equal to k_1/k_f [6], which is around 0.3 in the present case. In Figure 2, we show the Pr_M dependence of ϵ_K/ϵ_M for $\sigma = 1$ and 0. The simulations show that for both $\sigma = 1$ and 0, the ratio ϵ_K/ϵ_M scales with Pr_M,

$$\epsilon_K/\epsilon_M \propto \mathrm{Pr}_M^q, \tag{1}$$

but the exponent is not always the same. For $\sigma = 1$, we find $q \approx 2/3$ for both small and large values of Pr_M, while for $\sigma = 0$, we find $q \approx 0.6$ for $\mathrm{Pr}_M < 1$ with $\mathrm{Re} \approx 80$ and $q \approx 0.3$ for $\mathrm{Pr}_M > 1$ with $\mathrm{Re} \approx 460$. For large-scale dynamos ($\sigma = 1$), a similar scaling was first found for $\mathrm{Pr}_M \leq 1$ [1, 2], and later also for $\mathrm{Pr}_M \geq 1$ [4]. For $\mathrm{Pr}_M \leq 1$, this scaling was also found for small-scale dynamos [3], but now we see that for $\mathrm{Pr}_M \geq 1$ the slope is smaller.

Our results for $\mathrm{Pr}_M > 1$ are compatible with those of [7], who listed the kinetic and magnetic dissipation scales, $\ell_K = (\nu^3/\epsilon_K)^{1/4}$ and $\ell_M = (\eta^3/\epsilon_M)^{1/4}$, respectively, for their decaying and forced hydromagnetic simulations at different values of Pr_M. Computing the dissipation ratio from their Table 1 as $\epsilon_K/\epsilon_M = \mathrm{Pr}_M^3 (\ell_K/\ell_M)^{-4}$, we find that their data for non-helical decaying turbulence are well described by the formula $\epsilon_K/\epsilon_M \approx 0.6\,\mathrm{Pr}_M^{0.55}$. For non-helically forced turbulence with $0.01 \leq \mathrm{Pr}_M \leq 10$, their data agree perfectly with our fit $\epsilon_K/\epsilon_M \approx 0.4\,\mathrm{Pr}_M^{1/3}$ (red filled symbols in Figure 2). In their case, Re_M increases with Pr_M, but its value is generally much larger than our values for $\mathrm{Pr}_M < 1$. This suggests that the $1/3$ scaling occurs for large enough magnetic Reynolds numbers and that our steeper fit for $\mathrm{Pr}_M \leq 1$ and the mismatch at $\mathrm{Pr}_M = 1$ is a consequence of small values of Re_M.

We emphasize that in view of Figure 1, the fraction of energy that is being diverted to magnetic energy through dynamo action depends on the term $-\langle \boldsymbol{u} \cdot (\boldsymbol{J} \times \boldsymbol{B}) \rangle$, and that this must be equal to ϵ_M in the statistically steady state. This fraction is therefore ϵ_M/ϵ_T and we may call it the efficiency of the dynamo. Remarkably, Figure 2 shows that there is a Pr_M dependence of the dynamo efficiency both with and without helicity. The presence of helicity in the forcing function can lead to magnetic field generation at the largest scale of the system. It is therefore also referred to as a large-scale dynamo. Non-helical forcing leads to magnetic fields on scales that are typically somewhat smaller than the energy-carrying scale of the turbulent motions.

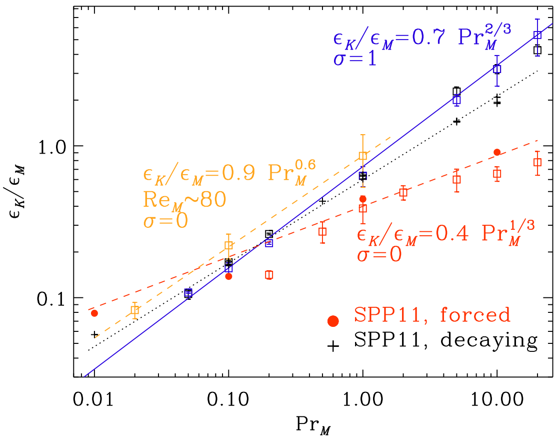

Figure 2. Dependence of the dissipation ratio ϵ_K/ϵ_M on Pr_M for large-scale dynamos (solid blue line) and small-scale dynamos (dashed orange and red lines). The red filled symbols and black plus signs correspond to the results of [7] for forced and decaying turbulence, respectively, referred to as SPP11 in the legend.

DISSIPATION RATIO IN ONE-DIMENSIONAL MODELS

We can produce a stationary state where the ram pressure of the flow from the left ($x \to -\infty$) can be balanced by the magnetic pressure of a magnetic kink when $b \to u_0$ for $x \to +\infty$ and $b \to 0$ for $x \to -\infty$. The resulting scaling in Figure 3 confirms Equation (1) with $q \approx 0.55$ for $\mathrm{Pr}_M > 1$ and $q \approx 0.95$ for $\mathrm{Pr}_M < 1$.

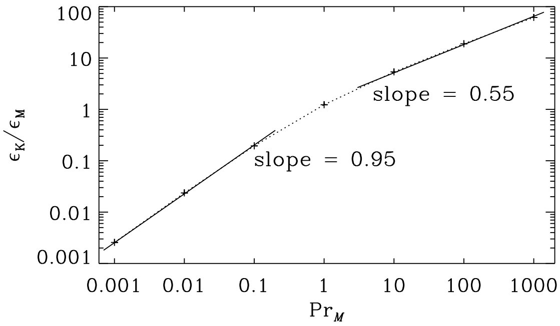

Figure 3. Magnetic Prandtl number dependence in the MHD model.

Here we find scalings that are broadly similar to those for turbulent large-scale dynamos as well as small-scale dynamos for $\mathrm{Pr}_M < 1$, namely a slope between 0.6 and 0.7. For $\mathrm{Pr}_M = 1$, the profiles of $b(x)$ and $u(x)$ are similar and resemble the $\tanh x/w$ profile of u in the passive scalar case. However, for both $\mathrm{Pr}_M \ll 1$ and $\gg 1$, the profiles of $b(x)$ and $u(x)$ become asymmetric, which is also the reason why we chose to integrate in a domain where $-x_- > x_+$. For small values of Pr_M, i.e., when $\eta \gg \nu$, the magnetic field begins to ramp up slowly and quite far away from $x = 0$. This leads to a corresponding decline of $u(x)$. On the other hand, for large values of Pr_M, the value of ν ($\gg \eta$) is so large that a certain imbalance of $u^2 + b^2 - u_0^2$ implies only a small slope in $u(x)$, so $|u'|$ must be small.

In the present work, he have extended earlier findings of a Pr_M dependence of the kinetic-to-magnetic energy dissipation ratio, ϵ_K/ϵ_M, to the regime of small-scale and large-scale dynamos for $\mathrm{Pr}_M > 1$ and at higher resolution than what was previously possible [4]. In most cases, our results confirm earlier results that for large-scale dynamos, the ratio ϵ_K/ϵ_M is proportionate to $\mathrm{Pr}_M^{0.6}$. Furthermore, we have shown that a similar scaling with Pr_M can be obtained for a simple one-dimensional Alfvén kink, where ram pressure locally balances magnetic pressure. Interestingly, in these cases kinetic energy dissipation is accomplished mainly by the irrotational part of the flow rather than the solenoidal part as in the turbulence simulations presented here. We note in this connection that the kinetic energy dissipation, which is proportional to $\langle 2\mathbf{S}^2 \rangle = \langle (\boldsymbol{\nabla} \times \boldsymbol{u})^2 \rangle + \langle \frac{4}{3} (\boldsymbol{\nabla} \cdot \boldsymbol{u})^2 \rangle$, has similar contributions from vortical and irrotational parts.

References

[1] P. D. Mininni, Inverse cascades and α effect at a low magnetic Prandtl number, Phys. Rev. E **76**, 026316 (2007).
[2] A. Brandenburg, Large-scale dynamos at low magnetic Prandtl numbers, Astrophys. J. **697**, 1206–1213 (2009).
[3] A. Brandenburg, Nonlinear small-scale dynamos at low magnetic Prandtl numbers, Astrophys. J. **741**, 92 (2011).
[4] A. Brandenburg, Dissipation in dynamos at low and high magnetic Prandtl numbers, Astron. Nachr. **332**, 51–56 (2011).
[5] N. E. L. Haugen, et al., Is nonhelical hydromagnetic turbulence peaked at small scales? Astrophys. J. **597**, L141–L144 (2003).
[6] A. Brandenburg, The inverse cascade and nonlinear alpha-effect in simulations of hydromagnetic turbulence, Astrophys. J. **550**, 824–840 (2001).
[7] G. Sahoo, P. Perlekar, R. Pandit, Systematics of the magnetic-Prandtl-number dependence of homogeneous, isotropic magnetohydrodynamic turbulence, New J. Phys. **13**, 013036 (2011).