# MEAN FIELD MODEL FOR TURBULENCE TRANSITION IN PLANE POISEUILLE FLOW

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Abstract In the pipe flow model of Dwight Barkley [1, 2], the main idea is to model pipe flow as an excitable, bistable medium. Using a one-dimensional FitzHugh-Nagumo-type reaction-advection-diffusion system with two variables the model captures qualitatively a surprising number of features of the turbulence transition in pipe flow. Motivated by this success, we here describe a derivation of a set of two 1+1-dimensional coupled differential equations for the closely related system of plane Poiseuille flow from the Navier-Stokes equation. The model contains terms for the production of turbulent kinetic energy, its transfer between the modes and its dissipation by viscous terms. The model shows a bifurcation to a non-trivial state and reflects some of the complex dynamics observed in direct numerical simulations.

## INTRODUCTION

The transition to turbulence in plane Couette flow, pipe flow and plane Poiseuille flow can be triggered by suitable perturbations in a range of Reynolds numbers where the laminar profile is still stable [3]. Considerable progress has been made through the identification of three-dimensional coherent structures [4, 5, 6, 7] that appear at flow rates below the ones where transition can be observed and which provide a scaffold of persistent structures around which the turbulence can form. The structures are remarkably consistent in appearance, are dominated by downstream vortices and can be rationalized within the self-sustaining cycle [8, 9]. They are exact solutions to the Navier-Stokes equation, and found numerically using suitably adapted and expanded Newton methods [10].
A key to the transition in these flows is the excitation and maintenance of transverse velocity components: without them, the flow becomes invariant in the downstream direction and decays. This observation provides the basis for our model, as well as for certain control strategies.

## TOTAL ENERGY BALANCE

We start with the incompressible Navier-Stokes equation and decompose the velocity field into a base flow $\mathbf{u}_0$ and a perturbation $\mathbf{u}'$. For the two components of the model we take the energy content of the downstream and span wise components. They are obtained by projecting the Navier Stokes equation for the perturbation onto the parallel $\mathbf{u}'_\parallel$ and perpendicular $\mathbf{u}'_\perp$ components, to arrive at a set of two coupled equations:

$$\partial_t \frac{\mathbf{u}_\parallel^2}{2} + \underbrace{\nabla\cdot(\mathbf{u}'\frac{\mathbf{u}_\parallel^2}{2}) + \partial_\parallel u_0 \frac{\mathbf{u}_\parallel^2}{2}}_{advection} + \underbrace{\mathbf{u}_\parallel(\mathbf{u}_\perp\cdot\partial_\perp)\mathbf{u}_0}_{production} = \underbrace{-\mathbf{u}_\parallel\cdot\nabla_\parallel p'}_{+transfer} + \underbrace{\frac{1}{Re}\mathbf{u}_\parallel\Delta\mathbf{u}_\parallel}_{dissipation_\parallel}$$

$$\partial_t \frac{\mathbf{u}_\perp^2}{2} + \underbrace{\nabla\cdot(\mathbf{u}'\frac{\mathbf{u}_\perp^2}{2}) + \partial_\parallel u_0 \frac{\mathbf{u}_\perp^2}{2}}_{advection} = -\nabla\cdot(\mathbf{u}'p') \underbrace{+\mathbf{u}_\parallel\cdot\nabla_\parallel p'}_{-transfer} + \underbrace{\frac{1}{Re}\mathbf{u}_\perp\Delta\mathbf{u}_\perp}_{dissipation_\perp}$$

Assuming a steady state and averaging over the entire volume gives a simple balance between energy production $P$, the transfer $T$ between the modes and the dissipations in the parallel and perpendicular components,

$$\begin{aligned} P &= T + \epsilon_\parallel \\ T &= \epsilon_\perp \end{aligned}$$

We can then express these terms as functions of the two variables $\pi := \langle\frac{\mathbf{u}_\parallel^2}{2}\rangle$ and $\sigma := \langle\frac{\mathbf{u}_\perp^2}{2}\rangle$ by parameter estimation from DNS data on fully turbulent flows, obtained using channelflow [11]. This results in a set of equations with two unknowns describing the nullclines $\dot\pi = 0$ (eq.(1)) and $\dot\sigma = 0$ (eq.(2)) of the system in a homogenous, steady state:

$$(a_P + \frac{b_P}{Re})\pi^{\frac{1}{2}}\sigma^{\frac{1}{2}} = (a_T + \frac{b_T}{Re})\pi\sigma^{\frac{1}{2}} + \frac{a_\epsilon^\parallel}{Re}\pi + b_\epsilon^\parallel \pi^{\frac{3}{2}} \tag{1}$$

$$(a_T + \frac{b_T}{Re})\pi\sigma^{\frac{1}{2}} = \frac{a_\epsilon^\perp}{Re}\sigma + b_\epsilon^\perp\sigma^{\frac{3}{2}}, \tag{2}$$

This set of equations has as fixed points the laminar state, $\pi = \sigma = 0$, and, as the Reynolds number increases, two new states created in a saddle node bifurcation at $Re \approx 1200$ (see the null clines in Figure 1).

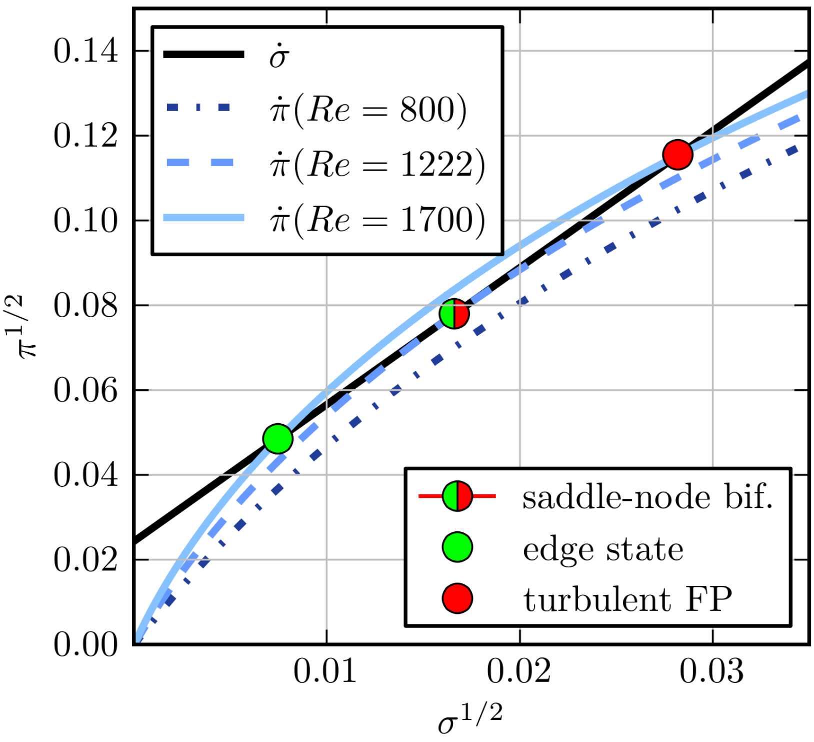


**Figure 1.** Bifurcation diagram for the case $\pi \neq 0$ and $\sigma \neq 0$. Shown are a linear graph mimicking the Reynolds number independent equation (1) representing $\dot{\pi} = 0$ and equation (2) representing $\dot{\sigma} = 0$ for different Reynolds numbers. As $\dot{\pi}$ is approaching $\dot{\sigma}$ for increasing $Re$, the nullclines intersect at $Re = 1222$. At this point, two new states are born in a saddle-node bifurcation, a stable, turbulent fixed point and an unstable fixed point separating laminar and turbulent basin of attraction, here denoted as "edge state".

In a broader context, the model is reminiscent of stochastic models studied, e.g., in [12, 13]. The relation between the two approaches and their implications for control will be discussed as well.

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