REYNOLDS NUMBER DEPENDENCE OF THE DIMENSIONLESS DISSIPATION RATE IN STATIONARY MAGNETOHYDRODYNAMIC TURBULENCE

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Abstract Results on the Reynolds number dependence of the dimensionless total dissipation rate C_ε are presented, obtained from medium to high resolution direct numerical simulations (DNSs) of mechanically forced stationary homogeneous magnetohydrodynamic (MHD) turbulence in the absence of a mean magnetic field, showing that $C_\varepsilon \to const$ with increasing Reynolds number. Furthermore, a model equation for the Reynolds number dependence of the dimensionless dissipation rate C_ε is derived from the real-space energy balance equation by asymptotic expansion in terms of Reynolds number of the second- and third-order correlation functions of the Elsässer fields $\boldsymbol{z}^\pm = \boldsymbol{u} \pm \boldsymbol{b}$. At large Reynolds numbers we find that a model of the form $C_\varepsilon = C_{\varepsilon,\infty} + C/R_{z^-}$ describes the data well, while at lower Reynolds numbers the model needs to be extended to second order in $1/R_{z^-}$ in order to obtain a good fit to the data, where R_{z^-} is a generalised Reynolds number with respect to the Elsässer field $\boldsymbol{z}^-$.
Keywords: magnetohydrodynamics, turbulence

INTRODUCTION

In the absence of a magnetic field it has been known for a long time that the total dissipation rate in forced and freely decaying homogeneous isotropic turbulence tends to a constant value with increasing Reynolds number following a well-known characteristic curve [1, 2]. Similar obervations have recently been reported in decaying magnetohydrodynamic (MHD) turbulence [3, 4], where it was found that the temporal maximum of the total dissipation in freely decaying turbulent hydromagnetic systems tends to a constant with increasing Reynolds number, using results from direct numerical simulations (DNSs). In this talk we present data from a series of DNSs of mechanically forced MHD turbulence on up to 1024^3 grid points using the standard pseudospectral method with full de-aliasing, showing that the dimensionless dissipation rate $C_\varepsilon \to const$ with increasing Reynolds number. Furthermore, we propose a model for the large Reynolds number behaviour of the dimensionless dissipation rate based on the energy balance equation for MHD turbulence in terms of Elsässer fields [5], and subsequently compare the model equation to DNS data [6].

DERIVATION OF THE MODEL EQUATION

For simplicity and in order to compare to results in the literature we consider the case of $Pr = \nu/\eta = 1$, where ν denotes the kinematic viscosity and η the resistivity. In order to obtain stationarity, we assume the system to be forced at the large scales. The real-space energy balance equation of MHD turbulence can be used in order to study the Reynolds number dependence of the dimensionless dissipation rate C_ε after appropriate non-dimensionalisation. Since we are interested in the total dissipation $\varepsilon = \varepsilon_{mag} + \varepsilon_{kin}$ there are two possible approaches: either formulating the energy balance in terms of velocity and magnetic field fluctuations $\boldsymbol{u}$ and $\boldsymbol{b}$; or in terms of Elsässer variables. Since $\eta = \nu$ one can set $\varepsilon^+ = \varepsilon - 2\partial_t H_c$, where ε^+ denotes the dissipation rate with respect to the Elsässer field $\boldsymbol{z}^+ = \boldsymbol{u} + \boldsymbol{b}$ and $H_c = \langle \boldsymbol{u} \cdot \boldsymbol{b} \rangle$ the cross helicity. For the stationary case $\partial_t H_c = 0$, and one obtains $\varepsilon = \varepsilon^+$. Thus the total dissipation rate can be described either by the energy balance equation for $\boldsymbol{z}^+$ [5], or by the sum of the energy balance equations for $\langle |\boldsymbol{b}(t)|^2 \rangle$ and $\langle |\boldsymbol{u}(t)|^2 \rangle$. The situation is different for the *dimensionless* dissipation rate C_ε. If we want to define an analogue to the hydrodynamic Taylor surrogate expression [7, 8], there are several choices of scales with which to non-dimensionalise. Since the total dissipation rate contains by definition magnetic and kinetic contributions, scaling it using magnetic and kinetic terms would be more appropriate than scaling it with the rms velocity U only. Therefore we propose to define the dimensionless dissipation rate with respect to Elsässer fields $C_\varepsilon = \varepsilon L_{z^+}/({z^+}^2 z^-)$, where L_{z^+} is the integral scale defined with respect to $\boldsymbol{z}^+$ and $z^\pm$ the rms values of $\boldsymbol{z}^\pm$. Using this definition we can now consistently non-dimensionalise the energy balance equation written in terms of $\boldsymbol{z}^+$, which reads for the stationary case

$$0 = \frac{\partial_r}{r^4}\left(r^4\left[\frac{3}{4}C^{++-}_{LL,L} - \frac{3}{8}B^{+-+}_{LL,L}\right]\right) + \frac{3}{4r^4}\partial_r\left(r^4\partial_r(\nu+\eta)B^{++}_{LL}\right) - I(r)\,, \tag{1}$$

where $C^{++-}_{LL,L}$, $B^{+-+}_{LL,L}$ and B^{++}_{LL} are the longitudinal correlation and structure functions corresponding to $\boldsymbol{z}^\pm$ and $I(r)$ is a scale-dependent energy input term. For scales much smaller than the forcing scale the energy input will be scale-independent, that is $I(r) = \varepsilon_W$. Here ε_W is the total rate of energy input, which must equal the total dissipation in the stationary case, hence

$$\varepsilon = \varepsilon_W = I(r) = \frac{\partial_r}{r^4}\left(r^4\left[\frac{3}{4}C^{++-}_{LL,L} - \frac{3}{8}B^{+-+}_{LL,L}\right]\right) + \frac{3}{4r^4}\partial_r\left(r^4\partial_r(\nu+\eta)B^{++}_{LL}\right)\,. \tag{2}$$

Introducing the nondimensional variable $\rho = r/L_{z^+}$ and non-dimensionalising the energy balance equation with respect to $z^{\pm}$ and L_{z^+} as proposed in the definition of C_ε, one obtains

$$C_\varepsilon = \frac{\varepsilon L_{z^+}}{{z^+}^2 z^-} = \frac{1}{\rho^4}\partial_\rho \left(\frac{3\rho^4 C^{++-}_{LL,L}}{4{z^+}^2 z^-} - \frac{3\rho^4 B^{+-+}_{LL,L}}{8{z^+}^2 z^-} \right) + \frac{\eta+\nu}{L_{z^+} z^-}\frac{3}{4\rho^4}\left(\rho^4 \partial_\rho \frac{B^{++}_{LL}}{{z^+}^2} \right) . \tag{3}$$

Note that the inverse of the coefficient in front of the dissipative term has the form $z^- L_{z^+}/(\nu+\eta)$, which is similar to a Reynolds number. Thus we introduce the generalised large-scale Reynolds number $R_{z^-} = 2z^- L_{z^+}/(\nu+\eta)$.
This already suggests a dependence of C_ε on $1/R_{z^-}$, however, the structure and correlation functions also have a dependence on Reynolds number. Therefore we consider asymptotic expansions of the dimensionless functions in inverse powers of R_{z^-}, which leads to the model equation

$$C_\varepsilon = C_{\varepsilon,\infty} + \frac{C}{R_{z^-}} + \frac{D}{R^2_{z^-}} + O(R^{-3}_{z^-}) . \tag{4}$$

COMPARISON TO DNS DATA

Figure 1 shows error-weighted fits of the model equation to DNS data. As can be seen, the model fits the data very well, provided we include terms of second order in R_{z^-}. For $R_{z^-} > 80$, it is sufficient to consider terms of first order in R_{z^-} only. The asymptote has been calculated to be $C_{\varepsilon,\infty} = 0.218 \pm 0.002$, where the error encompasses both the statistical standard error of the data and the error of the fit.

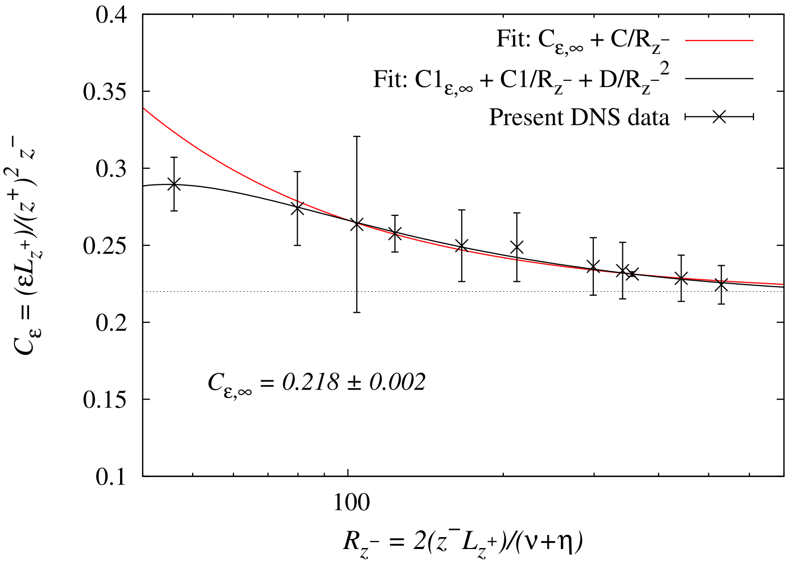

Figure 1. The expression given in equation (4) fitted to DNS data. The red line shows a fit to data for $R_{z^-} > 80$ to first order in $1/R_{z^-}$, the black line results from a fit using all data points and including terms up to second order in $1/R_{z^-}$.

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