EFFECTIVE AND ANOMALOUS DIFFUSION OF INERTIAL PARTICLES IN FLOWING FLUIDS

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Abstract We perform an analytical study of the inertial-particle dynamics in the limit of small but finite inertia, in incompressible flows, exploring their diffusion process. By means of a multiscale expansion, we analyse the particle effective diffusivity, and in particular its dependence on relative inertia, Brownian diffusivity, gravity, and particle-to-fluid density ratio (i.e. added mass). We obtain forced advection–diffusion equations for auxiliary quantities in the physical space, thus simplifying the problem from the full phase space to a system which can easily be solved numerically. In the case of parallel flows with power-law velocity spectra, we identify some cases of anomalous diffusion, according to the values of the exponents connected to the possible presence of long-range spatio–temporal correlations.

THE DYNAMICS OF INERTIAL PARTICLES

Let us consider a single, small, spherical inertial particle, of mass density $\rho_{\rm p}$ and radius r, subject to Brownian diffusivity (κ) and gravity ($\boldsymbol{g}$), and carried by a fluid with density $\rho_{\rm f}$ and kinematic viscosity ν. Let us focus on an incompressible flow $\boldsymbol{u}(\boldsymbol{x},t)$ and let us define two usual quantities, the density coefficient $\beta \equiv 3\rho_{\rm f}/(\rho_{\rm f}+2\rho_{\rm p})$ and the Stokes time $\tau \equiv r^2/3\nu\beta$. The particle position $\boldsymbol{X}(t)$ and covelocity $\boldsymbol{V}(t) \equiv \dot{\boldsymbol{X}}(t) - \beta\boldsymbol{u}(\boldsymbol{X}(t),t)$ evolve according to [1, 2]:

$$\begin{cases} \dot{\boldsymbol{X}}(t) = \boldsymbol{V}(t) + \beta\boldsymbol{u}(\boldsymbol{X}(t),t) \\ \dot{\boldsymbol{V}}(t) = -\dfrac{\boldsymbol{V}(t)-(1-\beta)\boldsymbol{u}(\boldsymbol{X}(t),t)}{\tau} + (1-\beta)\boldsymbol{g} + \dfrac{\sqrt{2\kappa}}{\tau}\boldsymbol{\eta}(t)\ , \end{cases} \tag{1}$$

where $\boldsymbol{\eta}(t)$ is the standard white noise mimicking thermal fluctuations, the added-mass term has been accounted for in a simplified way, and the action of the fluid has been modelled via the Stokes viscous drag. The Basset, Faxén, Oseen and Saffman corrections have been neglected, as well as any interaction with boundaries or other particles and any additional feedback on the fluid. The Fokker–Planck equation for the particle concentration $p(\boldsymbol{x},\boldsymbol{v},t)$ in the phase space reads:

$$\left\{\frac{\partial}{\partial t} + \frac{\partial}{\partial x_\mu}(v_\mu+\beta u_\mu) + \frac{\partial}{\partial v_\mu}\left[\frac{(1-\beta)u_\mu - v_\mu}{\tau} + (1-\beta)g_\mu\right] - \frac{\kappa}{\tau^2}\frac{\partial^2}{\partial v_\mu \partial v_\mu}\right\} p = 0\ . \tag{2}$$

Making use of the characteristic length and speed scales of the fluid flow, L and U respectively, it is customary to nondimensionalize the problem and to express the results in terms of three adimensional numbers: $\mathrm{St} \equiv \tau U/L$ (Stokes), $\mathrm{Pe} \equiv LU/\kappa$ (Péclet) and $\mathrm{Fr} \equiv U/\sqrt{gL}$ (Froude).

RENORMALIZED EDDY DIFFUSIVITY

Let us focus on a zero-mean flow field, steady or periodic in time (with period T) and cellular in space (with cell size L) [3]. Investigating the particle dynamics at large length and time scales, $\gg L$ and $\gg T$ respectively, we can use the multiple-scale technique [4, 5] and isolate a possible ballistic degree of freedom related to the terminal velocity $\boldsymbol{w}$ [6, 7], in order to be left with a purely diffusive large-scale behaviour:

$$\frac{\partial}{\partial \mathcal{T}}\mathcal{P}(\boldsymbol{\mathcal{X}},\mathcal{T}) = K_{\mu\lambda}\frac{\partial^2}{\partial \mathcal{X}^\mu \partial \mathcal{X}^\lambda}\mathcal{P}(\boldsymbol{\mathcal{X}},\mathcal{T})\ . \tag{3}$$

Here, $\mathcal{P}$ represents the particle concentration in the newly-introduced slow variables $\boldsymbol{\mathcal{X}} \equiv \epsilon\boldsymbol{x}$ and $\mathcal{T} = \epsilon^2 t$ (ϵ being the well-known scale-separation parametre), while the effective-diffusivity tensor results from an integration on the original fast variables (viz. on the cell periodicity in space and time) [8]:

$$K_{\mu\lambda} = \int_0^T \frac{\mathrm{d}t}{T}\int \mathrm{d}\boldsymbol{x}\int \mathrm{d}\boldsymbol{v}\ [v_\mu + \beta u_\mu(\boldsymbol{x},t) - w_\mu]\, q_\lambda(\boldsymbol{x},\boldsymbol{v},t)\ . \tag{4}$$

The vector $\boldsymbol{q}(\boldsymbol{x},\boldsymbol{v},t)$ satisfies the so-called cell problem, i.e. an auxiliary equation with the same operatorial structure of (2) (replacing p with $\boldsymbol{q}$) but with a right-hand side given by $[\boldsymbol{v}+\beta\boldsymbol{u}(\boldsymbol{x},t)-\boldsymbol{w}]\,p(\boldsymbol{x},\boldsymbol{v},t)$. The resolution of the full problem (which cannot be achieved analytically, except for some very simple flows) is prohibitive also numerically, because it involves $2d+1$ variables, d being the usual space dimension. However, in the limit of small inertia, the covelocity degree of freedom decouples from space and time, and we are left with a St-expansion of the particle diffusivity:

$$K_{\mu\lambda} = \mathrm{Pe}^{-1}\delta_{\mu\lambda} + \int_0^T \frac{\mathrm{d}t}{T} \int \mathrm{d}\boldsymbol{x}\ [u_\mu Q_\lambda + \mathrm{St}(1-\beta)u_\mu u_\lambda] + \mathrm{O}(\mathrm{St}^2)\ , \tag{5}$$

where the vector $\boldsymbol{Q}$ itself results from an expansion $\boldsymbol{Q}(\boldsymbol{x},t) = \boldsymbol{Q}_0(\boldsymbol{x},t) + \mathrm{St}\boldsymbol{Q}_1(\boldsymbol{x},t) + O(\mathrm{St}^2)$. Every coefficient of the latter expansion obeys forced advection–diffusion equations — obtained from solvability conditions — which can be solved analytically e.g. for the Kolmogorov shear flow, or numerically for more general cases (this is a relatively easy task, involving only $d+1$ variables). The equation for $\boldsymbol{Q}_0$ is

$$\left(\frac{\partial}{\partial t} + u_\mu \frac{\partial}{\partial x_\mu} - \mathrm{Pe}^{-1} \frac{\partial^2}{\partial x_\mu \partial x_\mu}\right) \boldsymbol{Q}_0 = \boldsymbol{u}(\boldsymbol{x},t)\ , \tag{6}$$

thus making the $O(\mathrm{St}^0)$ in $K_{\mu\lambda}$ coincide with the well-known tracer limit [9]. We also found the equation for $\boldsymbol{Q}_1$, but we do not report it here due to its length. We just mention that gravity may pop out at $O(\mathrm{St})$ for finite Fr (but not for parallel flows), or even down at the (tracer-like) order St^0 for finite $\boldsymbol{w}$.

ANOMALOUS DIFFUSION

If $u_\mu(\boldsymbol{x},t) = \delta_{\mu 1} u(x_2,\ldots,x_d,t)$ is a parallel flow (for us $d=2,3$), interesting analytical results can be found [10] for the presence or absence of anomalous diffusion, in case the energy spectrum (Fourier transform of the velocity correlation, in time and in the $d-1$ spatial directions perpendicular to the flow) has a power law form.

Let us start with time-independent flows such that the spectrum is a power of the modulus of the $(d-1)$-dimensional wave number,

$$\mathcal{U}(\vec{q},\omega) = q^\alpha \delta(\omega)\ , \tag{7}$$

and let us look for anomaly bounds as functions of the exponent α. By means of a simple balance in integrands, we find a divergence in the eddy diffusivity, i.e. the appearance of anomalous diffusion, for $\alpha < 3-d$ if $\boldsymbol{w}$ is infinitesimal with St, and for

$$\alpha < -1 \tag{8}$$

if sedimentation keeps finite for vanishing inertia. Such exact results at $O(\mathrm{St}^0)$ can be extended to the case of small but finite inertia by looking at the $O(\mathrm{St}^1)$ correction. We find that the former inequality does not change when adding this leading correction, which means that tracers and weakly-inertial particles have the same bound normality/anomaly, unless subleading terms induce spurious divergences or renormalizing summations which cannot be accounted for in the present perturbative scheme. On the other hand, notice that the latter inequality (8) is independent of d, and that the width of the normal region is increased by the finiteness of the settling process.

Now, we move to spectra of time-depemdent flows, of the type

$$\mathcal{U}(\vec{q},\omega) = q^\alpha \omega^\zeta\ , \tag{9}$$

where also the exponent ζ of frequency comes into play. Further studies of the vanishing-sedimentation case are left to a forthcoming paper, therefore here let us restrict ourselves to the situation where StFr^{-2} is a finite quantity. The bound is now a piecewise-straight line in the plane ζ vs. α, namely

$$\alpha < -3 \quad \cup \quad \zeta < -1 \quad \cup \quad \alpha + \zeta < -2\ . \tag{10}$$

Diffusion is thus normal only in a region in the upper right part. Comparing (10) with (8), we can assert that — unless the temporal coherence is very pronounced (strogly negative values of ζ) — the width of the normal region is increased by the introduction of a temporal dependence.

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