DIRECT VERSUS NOISE-INDUCED OPTIMAL TRANSITIONS FOR A MODEL SHEAR FLOW

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Abstract The transition from laminar to turbulent flow in parallel shear flows like pipe flow or plane Couette flow is not due to a linear instability of the laminar profile but requires finite amplitude perturbations. The perturbations have to be strong enough that they cross the boundary to the basin of attraction of the turbulent regime. We use a 2-dimensional model of the transition to turbulence to explore the underlying phase space structure for optimal perturbations with respect to three different criteria: the energy of the initial condition, the energy dissipation of the initial condition and the amplitude of noise in a noise-induced transition. We find that the optimal transition states are different, but that the scaling with Reynolds number is the same in all three cases. Implications for full simulations will be discussed.

INTRODUCTION

In flows where the laminar profile is linearly stable, finite amplitude perturbations are needed to provoke a transition to turbulence [1]. The state space of the system then divides up into at least three regions, the immediate basin of attraction of the laminar profile, the basin of attraction of the turbulent dynamics, and the boundary region inbetween. The boundary is formed by the stable manifold of a particular invariant flow state, the edge state: it is a saddle state that lies on this boundary and has only one unstable eigendirection [2]. Initial conditions need to cross the stable manifold of the edge state to be able to trigger turbulence [3, 4]. “Optimal” perturbations are those which can trigger turbulence and simultaneously are an extremum of a certain functional [5, 6, 7, 8]. The study of optimal states can contribute to a better understanding of the dynamics underlying the transition to turbulence and to the development of effective control strategies.

OPTIMA FOR DIRECT AND NOISY TRANSITIONS

In order to illustrate the concepts by 2-d state space plots, we use the two-variable model proposed by Baggett and Trefethen [9]. The model has one parameter R representing the Reynolds number, a laminar fixed point $x = y = 0$ that exists and is stable for all R, and a bifurcation for $R = \sqrt{8}$ in which symmetry related saddle-node states appear (with the node then representing the turbulent state). We then study optimal perturbations for the deterministic transition via suitably chosen initial conditions, and for transitions induced by noisy fluctuations.
The optimization criteria are the energy or the energy dissipation of the initial condition for the direct transition scenario. Geometrically, the optimal perturbation is found when the iso-contour of the respective functional touches the stable manifold of the edge-state. The optimal state then corresponds to the value of the functional up to which all initial conditions return to the laminar state and for slightly larger values the first initial conditions will reach the turbulent state. We find these optimals by a modified edge-tracking algorithm which minimizes the particular functional.
Figure 1 shows the phase space portrait of the model system for different Reynolds numbers together with the optimal initial conditions for the energy (which happens to be circular because of the isotropy of the coordinates in the energy). The graph shows that the optimal initial condition approaches the origin along the y-axis (representing vortices). Moreover, the stable manifold of the edge state becomes almost parallel to the x-axis (the streaks).
For the noisy case, we recall that within a linear approximation around the laminar fixed point, a Gaussian white noise added to each component results in a Gaussian probability density function (pdf) [10, 11]. This pdf is elongated along the streak axis, in the direction of the edge state. Figure 2 shows the pdf for the linearized flow (left) and for the fully nonlinear one (right). Comparing the two figures one notes that in the nonlinear case the pdf is stretched out across the stable manifold of the edge state, and these trajectories then undergo transition. The transition point is close to the edge state and has a considerable contribution from the streaks.

CONCLUSIONS

The study of transition in the 2-d model highlights the underlying state-space structure and points to the different transition mechanisms in deterministic and noisy situations [12]. We expect similar phenomena in plane Couette flow and other related shear flows.

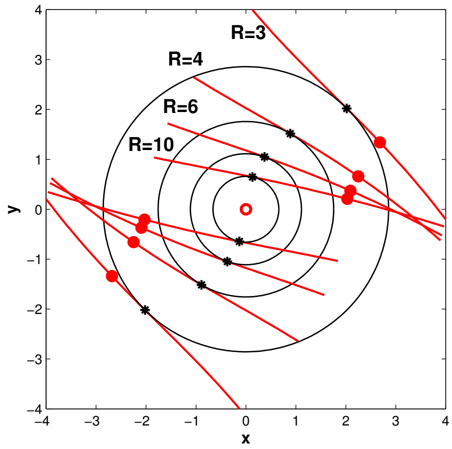

Figure 1. Optimal states in energy for different R. The open symbol in the middle is the laminar state. The full red symbols are the unstable edge states with the red lines indicating their stable manifolds. The black circles surrounding the laminar state indicate the states where the energy functional is minimal. The points of contact with the stable manifolds are marked by stars. One notes that as R increases the manifolds become more parallel to the x-axis, and the point of contact approaches the origin from the y-axis.

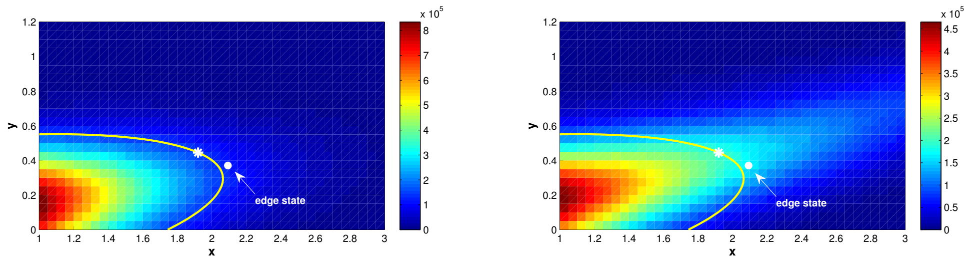

Figure 2. Left: Probability density for the linearized equations of motion with stochastic forcing for $R = 6$ in the region of phase space where the transition is expected to occur. It can be seen that the iso-contours of equal probability are of elliptical shape. The yellow line shows the optimal noise functional and the star indicates the point where it touches the stable manifold of the edge state. **Right:** Probability density as in the left figure, but for the full nonlinear equations with noise. Note that the iso-contours $p = const$ are stretched out towards the turbulent state and that they cross the stable manifold close to the indicated optimal state.

References

[1] S. Grossmann. The onset of shear flow turbulence. *Rev. Mod. Phys.*, **72**(2):603–618, 2000.
[2] J. D. Skufca, J. A. Yorke, and B. Eckhardt. Edge of Chaos in a Parallel Shear Flow. *Phys. Rev. Lett.*, **96**(17):174101, 2006.
[3] B. Eckhardt. Turbulence transition in pipe flow: some open questions. *Nonlinearity*, **21**(1):T1–T11, 2007.
[4] T. Kreilos and B. Eckhardt. Periodic orbits near onset of chaos in plane Couette flow. *Chaos*, **22**(4):047505, 2012.
[5] A. Monokrousos, A. Bottaro, L. Brandt, A. Di Vita, and D. S. Henningson. Nonequilibrium Thermodynamics and the Optimal Path to Turbulence in Shear Flows. *Phys. Rev. Lett.*, **106**(13):134502, 2011.
[6] C. C. T. Pringle, A. P. Willis, and R. R. Kerswell. Minimal seeds for shear flow turbulence: using nonlinear transient growth to touch the edge of chaos. *J. Fluid Mech.*, **702**:415–443, 2012.
[7] S. M. E. Rabin, C. P. Caulfield, and R. R. Kerswell. Triggering turbulence efficiently in plane Couette flow. *J. Fluid Mech.*, **712**:244–272, 2012.
[8] S. Cherubini and P. De Palma. Minimal perturbations approaching the edge of chaos in a Couette flow. *Fluid Dyn. Res.*, **46**, 2014.
[9] J. S. Baggett and L. N. Trefethen. Low-dimensional models of subcritical transition to turbulence. *Phys. Fluids*, **9**(4):1043–1053, 1997.
[10] B. Eckhardt, K. Marzinzik, and A. Schmiegel. Transition to turbulence in shear flows. In *A Perspective Look at Nonlinear Media*, pages 327–338. Springer, 1998.
[11] H. Risken. *The Fokker-Planck equation*. Springer Series in Synergetics. Springer, 3rd edition, 1996.
[12] M. Pausch and B. Eckhardt. Direct and noisy transitions in a model shear flow. *Submitted for publication*.