# INFLUENCE OF MAGNETIC DIFFUSION ON SHORT-WAVELENGTH MAGNETIC BUOYANCY INSTABILITY

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*Abstract:* Magnetic buoyancy instability (MBI) is believed to plays an important role in the evolution of magnetic fields in astrophysical objects. In the Sun large-scale toroidal magnetic field is probably generated in the tachocline – a thin region of strong radial shear, between radiative and convective zones. Observations indicate that this field rises the surface and create active regions. MBI is a probable mechanism for this phenomenon.
Magnetic and thermal diffusivity are parameters of solar plasma which could be important for dynamics of MBI. Here we consider preliminary case which include only magnetic dissipation. The singular perturbation methods are used to find difference with case without any dissipation.

## MATHEMATICAL FORMULATION

Following Mizerski et al. [1] we consider a plane layer of compressible, inviscid and isothermal fluid with neglected rotation in the presence of gravity and horizontal magnetic field decreasing with height. In this case, however, we assume nonzero magnetic diffusivity and its dimensionless form $\mathcal{U}_\eta$ will be our „small parameter" used in perturbation methods.

Main equations: the equations of motion, induction, continuity and perfect gas law takes the following dimensionless form:

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right) = -\mathcal{P}\nabla p - \rho\hat{\mathbf{e}}_z + \Lambda(\nabla\times\boldsymbol{B})\times\boldsymbol{B},$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{u} - \boldsymbol{B}(\nabla\cdot\boldsymbol{u}) + \mathcal{U}_\eta\nabla^2\boldsymbol{B},$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0,$$

$$p = \alpha\rho,$$

Basic state: in the region $0\leq z\leq 1$ with a horizontal depth-dependent magnetic field $\boldsymbol{B}=\bar{B}(z)\hat{\mathbf{e}}_x$, the pressure and density of the basic state are determined by equations:

$$\mathcal{P}\frac{d\bar{p}}{dz} = -\frac{\Lambda}{2}\frac{d\bar{B}^2}{dz} - \bar{\rho} \qquad \text{and} \qquad \bar{p} = \alpha\bar{\rho}.$$

Perturbations: we assume following form of two-dimensional perturbations to basic state, varying in the directions perpendicular to basic magnetic field:

$$\boldsymbol{u} = \big(0, v(z), w(z)\big)e^{\sigma t + iky},$$

$$\boldsymbol{b} = (b_x(z), 0, 0)e^{\sigma t + iky},$$

$$p = \tilde{p}(z)e^{\sigma t + iky}, \qquad \rho = \tilde{\rho}(z)e^{\sigma t + iky}.$$

Linearization: substituting basic state and perturbations into main equations and linearizing them, leads to the set of equations for MBI. Due to the presence of diffusive terms, the reduction of this set of equation is much more complicated than in the case with $\mathcal{U}_\eta=0$. However, it is possible to reduce it to one fourth-order ordinary differential equation for the vertical velocity $w$, with the instability growth rate $\sigma$ determined as the eigenvalue of the problem and with the boundary conditions on the vertical velocity at the horizontal boundaries.

Since we are interested in short-wavelength MBI we can associate wavenumber $k$ with dimensionless magnetic diffusivity $\mathcal{U}_\eta$ (which we assume to be small) in relation:

$$k \sim \mathcal{U}_\eta^{-n} \gg 1$$

In the final MBI equation limit $\mathcal{U}_\eta \rightarrow 0$ is singular, with the coefficient of the highest derivative tending to zero. This suggest that the eigenmodes become localized in the vicinity of $z_0$ point when $\mathcal{U}_\eta$ becomes small and we may use boundary layer method. Analysis is similar to presented in Mizerski et al. [1], however, we need to include aforementioned relation. A local (scaled) length variable is introduced:

$$\xi = \frac{z - z_0}{\delta}, \;\; \delta \sim \mathcal{U}_\eta^m \sim k^{-m/n},$$

where $\delta(\mathcal{U}_\eta)$ is a measure of the thickness of the boundary/internal layer. We expand all functions of $\xi$ (in the vicinity of $z_0$) and the growth rate $\sigma$ in powers of $\delta$:

$$\sigma = \sigma_0 + \delta\sigma_1 + \delta^2\sigma_2 + \ldots$$

We also define function:

$$\sigma^2(z) = \frac{\Lambda\bar{B}^2}{\bar{\rho}F}\left(\frac{1}{H_\rho} - \frac{1}{H_B}\right), \quad \text{where} \quad F = \mathcal{P}\alpha + \frac{\Lambda\bar{B}^2}{\bar{\rho}}, \;\; \frac{1}{H_\rho} = \bar{\rho}^{-1}\frac{d\bar{\rho}}{dz}, \;\; \frac{1}{H_B} = \bar{B}^{-1}\frac{d\bar{B}}{dz}.$$

## RESULTS

We need to find only possible distinguished limits (determined by values of $m$ and $n$, and relation between them). Hence there are following results for the most unstable mode, for the case where the $0<z_0<1$ is a maximum of a $\sigma^2(z)$ function (thus $\sigma_1=0$):

Leading order $\sigma_0$:

$0 < n < \frac{1}{2} \Leftrightarrow 1 < k < \mathcal{U}_\eta^{-1/2}$. Distinguished limit: $m = n \Rightarrow \delta \sim \mathcal{U}_\eta^n \sim k^{-1}$.

$\sigma_0 = \sigma^2(z_0)$. Case same as Mizerski et al. (2013) – no influence of magnetic diffusivity on $\sigma_0$

Second order $\sigma_2$:

$n = \frac{1}{3} \Leftrightarrow k \sim \mathcal{U}_\eta^{-1/3}$. Distinguished limit: $m = 1/6 \Rightarrow \delta \sim \mathcal{U}_\eta^{1/6} \sim k^{-1/2}$

Final equation reduces to: $\frac{\sigma_0^2}{\delta^4 k^2}\frac{d^2 w(\xi)}{d\xi^2} = \left[\frac{k^2\mathcal{U}_\eta\sigma_0\mathcal{P}\alpha}{\delta^2 F} + 2\sigma_0\sigma_2 - \frac{1}{2}\xi^2\frac{d^2}{dz^2}\sigma^2(z_0)\right]w(\xi)$

Solution: $\sigma_2 = -\frac{k^3\mathcal{U}_\eta\mathcal{P}\alpha}{2F} - \frac{1}{2}\sigma_0^2\left(-\frac{d^2}{dz^2}\sigma^2(z_0)\right)^{1/2}$

Thus we have influence of magnetic diffusivity on growth rate $\sigma$ at second order:
1) magnetic diffusivity decrease growth rate of the most unstable mode,
2) magnetic diffusivity establish the magnitude of wavenumber $k$ and the relation is:

$$k \sim \mathcal{U}_\eta^{-1/3}$$

## References

[1] K. A. Mizerski, C. R. Davies and D. W. Hughes. Short-Wavelength Magnetic Buoyancy Instability. *The Astrophysical Journal Supplement Series* **205**: 16-28, 2013.