REVERSIBILITY IN THE 3D INERTIAL TURBULENT CASCADE

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Abstract The inviscid nature of the of the inertial range of the turbulent energy cascade suggests that it should be reversible, and that reverse cascade effects should remain in normal turbulence. Using a reversible LES model, we study some properties of the direct and reverse cascades in isotropic turbulence. The reverse cascade is fairly stable and resilient to perturbations. The study of the Lyapunov exponents and eigenvectors of both cascades allows us to compute a 'viscous' limit below which the system is enslaved to larger scales, and to characterize the most unstable solutions of the forward cascade. They appear to correspond to stretched vortices.

The inertial range is by definition not affected by viscosity or by energy injection processes, and it is reasonable to expect that the dynamics of this range of scales is reversible. In practice, this is prevented by the irreversible dissipative range, but it is known that reversibility can be achieved using some LES models[1]. We take advantage of this possibility to study the reversible chaotic behaviour of the cascade itself. A Dynamic Smagorinsky LES[2, 4] is used to simulate the decay of isotropic turbulence in a triply-periodic box without molecular viscosity. The numerical integration is alias-free Fourier in space and 3rd-order Runge–Kutta in time stepping, with maximum wavenumber $k_{max} = 42$. Statistic are calculated for an ensemble of up to 2000 realisations, computed on GPUs. After the system has decayed for a while, the sign of the velocity field is reversed, $\vec{u} \to -\vec{u}$, after which the system evolves back, recovering energy and other turbulent quantities (solid lines in Fig. 1). During the forward evolution, the energy cascades from large to small scales to be dissipated by the model, whereas in the inverse evolution, energy is injected by the model and travels up to the large scales. Not only the energy, but the full flow field is restored. Since the model acts mostly in the small scales, this behaviour shows that not only the model is reversible, but also the inertial scales, where the model is negligible.

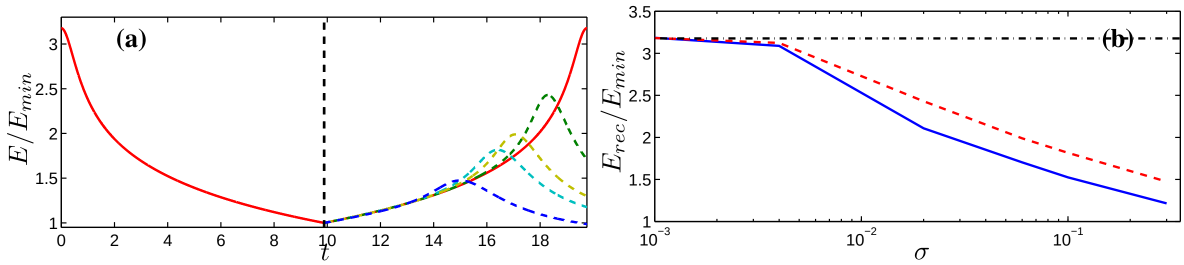

Figure 1. (a) Energy evolution in time over minimum energy. Vertical dashed line marks reversal time. Solid red line, exact reversal; dashed, approximate reversal for $k > k_{max}/2$. From blue to green: $\sigma = [0.3, 0.1, 0.06, 0.02]$. (b) Energy recovered for different standard deviations of the phase perturbations at reversal. Dashed, small scale ($k > k_{max}/2$); solid, large scale ($k < k_{max}/2$).

Full recovery only takes place for the particular initial condition in which the velocities are exactly reversed. We check the robustness of the reversibility by perturbing that condition. In those experiments, shown by dashed histories in figure 1a, a random phase shift is introduced in some velocity Fourier modes. The new transformation is $\vec{\hat{u}} \to -\vec{\hat{u}} \exp(\mathrm{i}2\pi\phi)$, where ϕ is a Gaussian random variable with zero mean and standard deviation σ. Note that this procedure does not change the energy of the system. Small ($k > k_{max}/2$) and large ($k < k_{max}/2$) scales are perturbed separately, and the amount of energy recovered is given in Fig. 1b as a function of σ. Not surprisingly, the energy recovered decreases with increasing perturbations. Also, the system is most sensitive to perturbing the large-scale , which contain most of the energy (95%), but that difference is not large. The energy in the perturbations is proportional to the energy of the perturbed modes, and it could be expected that perturbing the small scales would not be very effective. As we will see later, the reason that this is not so is that the small scales are the most unstable in the reverse cascade.

It is striking that even for strong perturbations of the large scales, $\sigma = 0.1$, the energy at reversal is doubled. The energy of the difference between those initial conditions and the exactly sign reversed ones is about 40% of the total. This shows that, although this inverse cascade is unlikely to be found in real situations, it is resilient and dynamically robust.

Further analysis of the stability and chaotic nature of the system is carried out by calculating the most unstable short-time Lyapunov exponent (STLE), which measures the divergence of nearby trajectories in phase space. Both for the inverse and the direct phases of the evolution, a small random perturbation δ is introduced in the initial conditions, $u' = u + \delta$, and both flows are evolved. Initially, δ is not aligned to the most unstable direction, but that 'eigenvector' eventually prevails. To allow this alignment while avoiding nonlinear effects, the magnitude of $|\delta|$ is periodically rescaled to some small value without changing its direction [3]. The STLE is calculated as $\lambda(t) = 1/\Delta t \log(|\delta(t + \Delta t)|/|\delta(t)|)$.

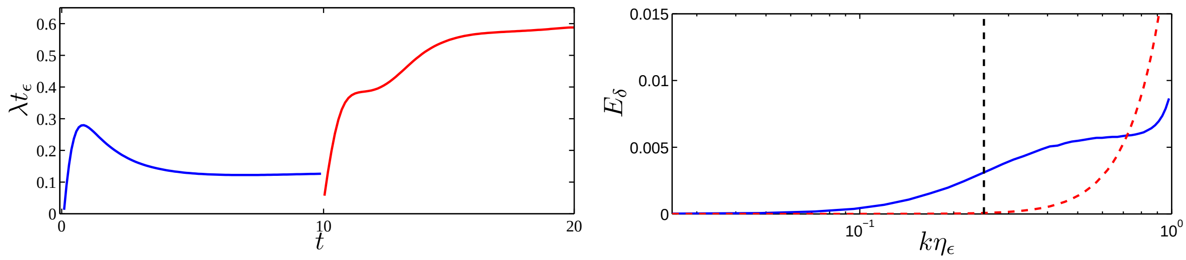

Figure 2. (a) STLE evolution in time for direct (blue) and inverse (red) cascade. (b) Mean premultiplied spectra of perturbation vector δ for direct (blue solid) and inverse (red dashed) cascade. Dashed black line is $k\eta_\epsilon = 0.25$, the limit for the viscous boundary.

Figure 2a shows the evolution of the STLEs for the direct and reverse cascades. They are normalized with a dissipative time, t_ϵ, obtained from the dissipation ϵ and a spatially averaged eddy viscosity obtained from ϵ and the averaged enstrophy, $\nu_\epsilon = \epsilon/\omega^2$. A pseudo-Kolmogorov length scale can then be formed that plays the role of a viscous length for the model. Figure 2a shows the convergence of the exponents, and hopefully their eigenvector, to their asymptotic states. As expected, the system is chaotic in both directions, with positive STLEs. The value $\lambda t_\epsilon = 1.25$ matches the values found in [3] for a DNS turbulent channel. The STLE of the reverse evolution is different from the direct one. Because of the time reversal, it represents the most contractive or stable exponent in the direct case.

If we interpret the STLE as the inverse of a time scale, and assume inertial relations, the largest forward exponent can be associated to the length scale, $l_\lambda = (\epsilon/\lambda^3)^{1/2}$, that marks the small-scale boundary for the chaotic behaviour of the system. The direct exponent corresponds to the eddy turnover time of the smallest scale where the chaotic behaviour of the system is not constrained by the model. Smaller scales contract in phase space and remain coupled to the dynamics of the larger ones. In pseudo-Kolmogorov units, $l_\lambda = 25\eta_\epsilon$ or $k\eta_\epsilon = 0.25$, and it is interesting that a similar limit was found in [5] for DNS.

Although the Lyapunov vectors selected by the STLE procedure are not real eigenvectors, because they change in times and differ for different trajectories, it is interesting that their energy spectrum is very nearly the same in most of our experiments. The converged mean spectra of the direct and reverse δ are shown in Fig. 2b. In the direct case, the perturbation is concentrated below $l_\lambda < 25\eta_\epsilon$, implying that a perturbation with that spectrum is the most unstable feature of the flow, and dynamically behaves as a block. Although the shape of this perturbation field in physical space is different for each experiment, not only its spectrum but its statistics are fairly stable. It will be shown in the final paper that it appears to correspond to a stretched vortex.

The spectra of the most unstable reverse δ is shown as a dashed line in Fig. 2b. It represents the perturbation that is damped the fastest in the direct cascade, and shows how the inverse cascade is more sensitive to perturbations in the small scales. It is interesting that its support coincides with the constant part of the direct eigenvector, suggesting that the most unstable forward perturbation is enslaved below its dissipative limit because the effect of the model is to contract it onto the attractor. It is then clear how $k\eta_\epsilon = 0.25$ sets the limit from which the model has an important effect in the dynamics. Smaller scales are bound together by the model, behave as one unique block and are at the same time driven by the effect of the dynamics of scales larger than $k\eta_\epsilon = 0.25$.

In summary, we have shown that the reverse energy cascade is statistically improbable but quite robust. Even when strongly perturbed, the reverse flux of energy survives, suggesting that it may play an important role in local regions of natural cascades. Furthermore, time symmetry allows us to study the largest and smallest short-time Lyapunov exponents. The former is associated to the expanding chaotic features of the flow, whereas the latter is related to the contractive nature of the LES model. The time scale of the largest STLE sets the limit of a 'viscous' boundary, below which the model acts as a constraint to chaos and the dynamics are enslaved to larger scales.

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