TRUNCATION OF SCALES BY RELAXATION

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Abstract This paper is about a relaxation model for large-eddy simulation of turbulent flow that truncates the too small scales of motion by making sure that they do not get energy from the larger eddies. To verify that a box filter is introduced and the relaxation parameter is determined in such a way that the production of small, box-fitting scales is counteracted by the modeled dissipation. This dissipation-production balance is worked out with the help of Poincaré's inequality, which results in a relaxation model that depends on the invariants of the velocity gradient. This model is discretized and equipped with a Schumann filter. It is successfully tested for isotropic turbulence as well as for turbulent channel flow.

To model the dynamics of spatially filtered turbulent flows, the Navier-Stokes (NS) equations are written as

$$\partial_t u \,+\, \nabla\cdot\overline{(\overline{u}\otimes\overline{u})} \,-\, \nu\,\nabla\cdot\nabla u \,+\, \nabla p \,=\, \nabla\cdot\Big(\overline{(\overline{u}\otimes\overline{u})} - u\otimes u\Big) \tag{1}$$

where the filtered velocity field is denoted by $\overline{u}$. If we replace the right-hand side by a 'model', we get

$$\partial_t v \,+\, \nabla\cdot\overline{(\overline{v}\otimes\overline{v})} \,-\, \nu\,\nabla\cdot\nabla v \,+\, \nabla\pi \,=\, -\nabla\cdot\overline{\tau(\overline{v})} \tag{2}$$

where the variable name is changed from u to v (and p to π) to stress that the solution of Eq. (2) differs from that of Eq. (1), because the model is not exact. Basically, the idea is that the large scales of motion remain virtually unchanged, whereas the tail of the modulated spectrum (i.e., the spectrum of v) falls of much faster than the NS-spectrum (i.e., the spectrum of u). This idea can be formalized by introducing a second filter. If we choose the box filter

$$\widetilde{v} = \frac{1}{|\Omega_\delta|}\int_{\Omega_\delta} v(x,t)\,dx, \tag{3}$$

for that purpose, then the larger eddies are described by the part of the fluid motion with velocity $\widetilde{v} \approx \widetilde{u}$. The residual velocity field $v - \widetilde{v}$ does not have any physical significance; it is only used to shorten the energy spectrum. Finally it may be remarked that the length of the LES-filter that is introduced in Eq. (1) is supposed to be less than (or at most equal to) the diameter δ of the box filter (3) that is used to verify that the spectrum is truncated properly.

The right-hand side of Eq. (1) does not dissipate energy, but transfers it (on average) towards smaller scales of motion that can dissipate energy at a higher rate. Here, we do not try to model the transport itself, but only just the net effect thereof. So the model should strengthen the dissipation (without producing smaller scales of motion, of course). To that end, we study the model/regularization introduced by Stolz *et al.* [1]. They used the relaxation operator

$$\nabla\cdot\tau(v) \,=\, \chi(v - \widetilde{v}) \tag{4}$$

which aims precisely to truncate the small scales of motion by dissipating their energy. The attractive feature of their relaxation method is that no (explict) use is made of a differential operator.

In this paper, the relaxation parameter χ is determined from the requirement that the production of any details of size smaller than δ by the convective nonlinearity is counteracted by the modeled dissipation. To develop this dissipation-production balance, we make use of Poincaré's inequality

$$\int_{\Omega_\delta} ||v-\widetilde{v}||^2\,dx \;\le\; C_\delta \int_{\Omega_\delta} ||\nabla v||^2\,dx,$$

The residual field $v' = v - \widetilde{v}$ contains eddies of size smaller than δ. The relaxation model must keep them from becoming dynamically significant. Poincaré's inequality shows that this can be achieved by damping the velocity gradient. According to Eq.(2) the $L^2(\Omega_\delta)$ norm of ∇v is governed by

$$\frac{\mathrm{d}}{\mathrm{dt}}\int_{\Omega_\delta} \tfrac{1}{2}||\nabla v||^2\,dx = \int_{\Omega_\delta} \nabla\left(\nu\,\nabla\cdot\nabla v \,-\, \nabla\pi\right) : \nabla v\,dx - \int_{\Omega_\delta} \nabla\nabla\cdot\tau(\overline{v}) : \nabla\overline{v} - 3R(\overline{v})\;dx - \int_{\partial\Omega_\delta} Q(\overline{v})\overline{v}\cdot n\,ds$$

where the production and transport of fine flow details are expressed in terms of the invariants $R(v) = -\frac{1}{3}\nabla v : \nabla v \nabla v$ and $Q(v) = \frac{1}{2}\nabla v : \nabla v$ of the velocity gradient; n is the outward-pointing normal vector to the boundary $\partial\Omega_\delta$ of Ω_δ. Thus we see that the upperbound given by Poincaré's inequality dissipates at its natural rate, that is the convective contribution to the evolution of the $L^2(\Omega_\delta)$ norm of ∇v is properly balanced by the modeled dissipation, if

$$\int_{\Omega_\delta} \nabla\nabla\cdot\tau(\overline{v}) : \nabla\overline{v}\,dx \;=\; 3\int_{\Omega_\delta} R(\overline{v})\,dx \;-\; \int_{\partial\Omega_\delta} Q(\overline{v})\,\overline{v}\cdot n\,ds \tag{5}$$

The scale truncation condition associated with the relaxation model is obtained by substituting Eq. (4) into Eq. (5):

$$\chi \;=\; \frac{3\int_{\Omega_\delta} R(\overline{v})\,dx \;-\; \int_{\partial\Omega_\delta} Q(\overline{v})\,\overline{v}\cdot n\,ds}{2\int_{\Omega_\delta} Q(\overline{v})\,dx} \tag{6}$$

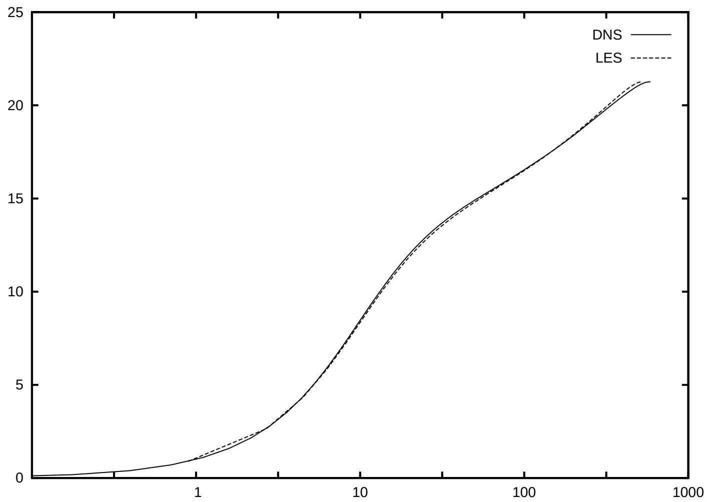
where the relaxation parameter is taken constant in Ω_δ. It depends explicitly on the LES-filter that is applied to suppress the high frequencies that are generated by the nonlinear, convective part of Eq. (2). Further, it is clear that the subfilter part $v - \overline{v}$ is not needed to calculate the relaxation parameter. In other words, the model can be evaluated directly, i.e., without applying any form of deconvolution to the LES-filter; so any difficulties associated with the deconvolution procedure are circumvented. If χ is negative, the small box-fitting scales transfer energy to the larger eddies Since these small scales of motion have no physical significance, χ is set to zero if Eq. (6) yields a negative value (i.e, χ is clipped).

When the explicitly filtered equation (2) is discretized in space, the low-pass characteristics of the discrete operators effectively act as a filter too. This numeric filter will inevitably interact with an explicit LES filter. So, at the discrete level, the effective filter is not so clear, unless we use a Schumann filter [2]. Therefore, we take

$$\overline{v} = \frac{1}{|\Omega_h|}\int_{\Omega_h} v(x,t)\,dx,$$

where Ω_h denotes the computational cell. So as in Schumann's approach the spatial discretization of the convective term defines the filter. Here we use a second-order, symmetry-preserving finite-volume method, see Ref. [3], e.g. In one spatial dimension, the convective derivative is approximated to second-order accuracy as $(v_{i+1} - v_{i-1})\,/\,(2h)$; hence, it does not see a point-to-point oscillation. Therefore we take $\delta = 2h$ [4]. In 1D the box filter (3) is approximated by $\widetilde{v}_i = \frac{1}{2}v_i + \frac{1}{4}\,(v_{i+1} + v_{i-1})$. This discretization rule is also applied to the Ω_δ-integrals in Eq. (6). It may be stressed that we approximate all integrals using the trapezoidal rule with constant coefficients, even if the grid is non-uniform, since the point-to-point mode must be an integral part of the residue of the discrete box filter. The invariants $Q(\overline{v})$ and $R(\overline{v})$ are computed from the discrete velocity gradient, where the gradient is discretized as in the convective term, that is over a distance $2h$. Thus, the length scale δ enters the discretizaton.

The performance of the proposed discrete scale-truncation model has been investigated for isotropic turbulent and turbulent channel flow. As an example results for turbulent channel flow (Re$_\tau$ = 590 [5]) are shown below. The computational grid consists of 64^3 points. More results will be shown at ETC15.



References

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