NEUTRAL STABILITY OF THE FLOW IN A TOROIDAL PIPE

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Abstract This work is concerned with the numerical investigation of the linear stability properties of the viscous, incompressible flow inside a toroidal pipe. A Hopf bifurcation is found and tracked in phase space, showing that the flow is modally unstable even at extremely low curvatures. The bifurcation and the eigenfunctions associated with it are analysed as a function of the two parameters governing the flow, *i.e.* the Reynolds number, *Re*, and the curvature, δ. For all curvatures, the critical Reynolds number is found to be about 3000.

MOTIVATION AND BACKGROUND

The stability of the flow inside a straight pipe – one of the most canonical configurations in fluid dynamics – has been one of the most studied problems of the past century. Despite this fact, no studies on the stability of curved pipes are known to the authors. In this work, as a first step, we analyse a toroidal pipe, which differs from a straight pipe by the addition of one single parameter, the curvature, and we study its influence on the stability properties of the flow. Bent pipes are omnipresent: going from industry where they are employed in devices ranging from heat exchangers, to exhausts, to junctions between pieces of straight pipe (for a review see Ref. [13]), to biology as, for instance, in the respiratory and circulatory systems. Understanding and quantifying the mechanisms that lead to transition to turbulence in bent pipes would thus aid in the design of such devices and in the understanding of systems of pipes involving bends.

Despite its relatively simple configuration, the flow inside a toroidal pipe cannot be described analytically; this has lead to the use of approximations to derive a closed-form solution to the Navier–Stokes equations in this geometry. The first solution has been proposed by Dean [3] who treated the problem in the limit of vanishing curvature and computed the first steady state solutions. He discovered the presence of two symmetric, counter-rotating vortices, later named in his honour, and introduced a unified scaling parameter, the Dean number, accounting for both the Reynolds number and the curvature. Following Dean, several other authors have employed the small curvature approximation in order to characterise this flow from a numerical point of view [9, 4, 12], for a review see Ref. [1]. It is only recently that studies considering *Re* and δ as two separate parameters have started to appear [5, 10]; we show that this is the right approach and that a single scaling parameter is not sufficient for a description this flow. Experimental works mostly consist in the study of the flow inside a helical pipe in the low-pitch limit [7, 11, 6, 14, 2]; this is due to the difficulty of forcing the fluid inside a closed torus. Only one experiment inside a toroidal pipe is known to the authors [8]. Of the cited works only two [5, 8] present results on the stability properties of this flow, but they are limited to only three values of δ.

RESULTS AND OUTLOOK

The problem is solved numerically with an in-house developed nonlinear solver based on the Finite Elements Method. Figure 1 shows an example of the steady, nonlinear base flow used for the subsequent stability calculations, obtained by Newton's method. A first computation of eigenvalue spectra for a grid of points in the parameter space revealed the presence of unstable eigenvalues, thus establishing that the flow is indeed modally unstable, in contrast with the flow inside a straight pipe. Each spectrum presents a pair of complex conjugate eigenvalues as least stable modes, indicating that the flow inside a toroidal pipe encounters a Hopf bifurcation as first instability on its route to turbulence. Beyond this bifurcation the steady state solution becomes unstable and the system is attracted by a stable limit cycle, in agreement with what has been observed both in the numerical work by Di Piazza & Ciofalo [5] as well as in the experimental work by Kühnen *et al.* [8]. The eigenspectra also revealed that the bifurcation is not identified by a single isolated mode: different families of modes and a few isolated eigenvalues take turns as critical modes, this behaviour is depicted in figure 2*a* in which every line corresponds to the neutral curve of one mode. Although the critical mode is always identified by a pair of complex conjugate eigenvalues, the identity of the pair, its wave number, frequency and symmetry properties are dependent on the curvature, creating a complex picture in the phase space; the neutral curve for this flow is thus composed by considering the envelope of these curves. Figure 2*b* shows an example of an eigenfunction at criticality (zero growth rate) for $\delta = 0.01$ and $Re = 4257$; in this case the mode is anti-symmetric. A minimum of one nonlinear DNS per family has been employed to validate the results and to assess the importance of the linear instability for this flow. In particular, the critical Reynolds number and the behaviour of the flow after the onset of the instability have been compared to those predicted by the stability analysis with excellent agreement.

The final contribution will include a detailed analysis of the steady states (base flow) for the whole parameter range, and an in-depth characterisation of the modes responsible for the instability as a function of the curvature. The results are currently limited to $\delta \in [0.001, 1]$ but an extension to zero curvature is underway.

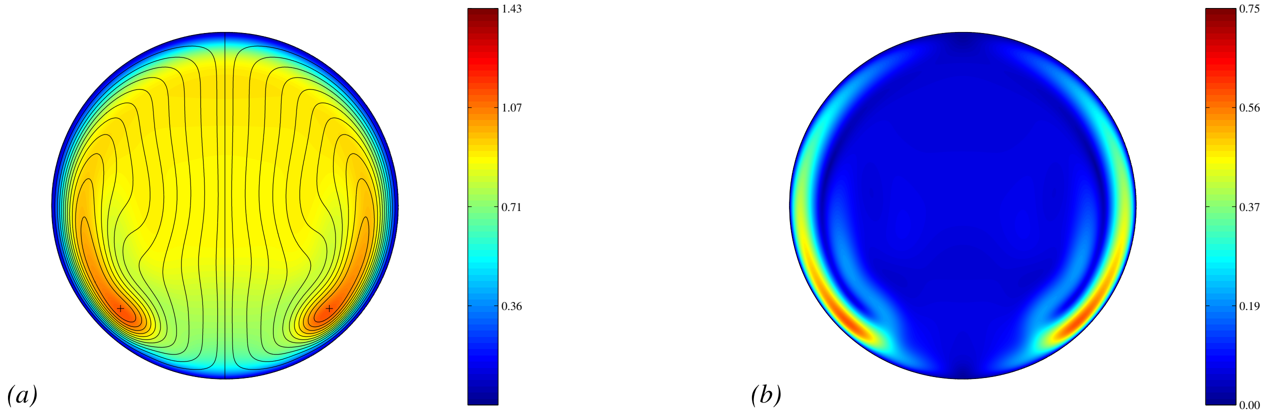

Figure 1. Steady state solutions for $\delta = 0.9$, $Re = 800$ corresponding to $De = 759$. *(a)* depicts the magnitude of the axial velocity and contours of the in-plane streamfunction. *(b)* shows the magnitude of the in-plane velocity.

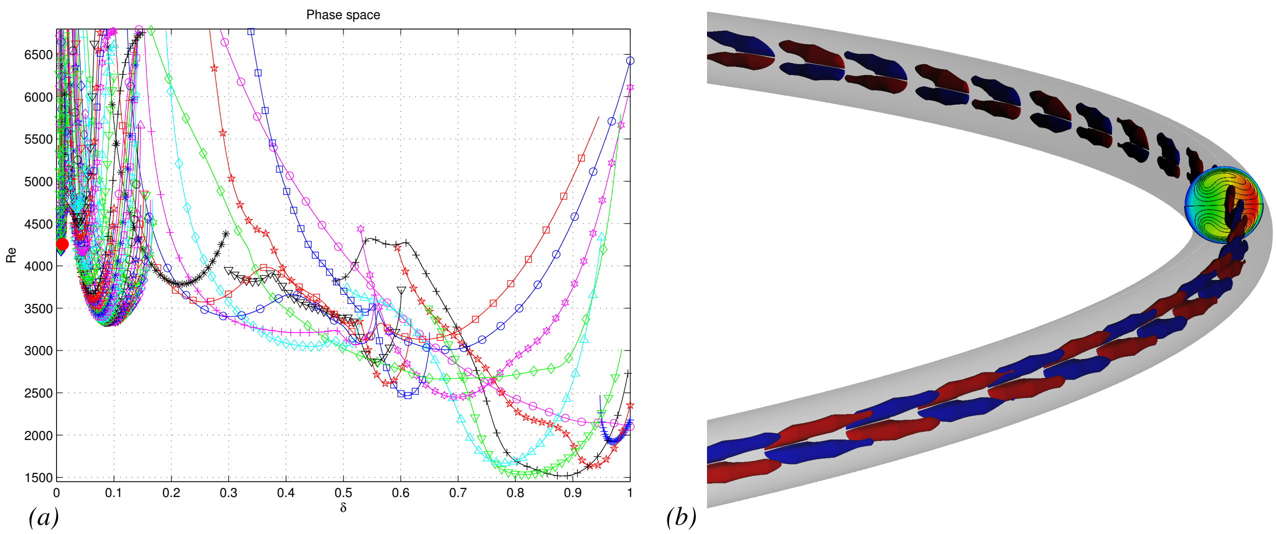

Figure 2. *(a)* Neutral curves in the $\delta - Re$ plane. Each line corresponds to the neutral curve of one mode; the envelope of the lines constitutes the neutral curve for the flow. It is possible to discern five families, each composed by a handful of modes with common characteristics. *(b)* One eigenmode belonging to the leftmost family; $\delta = 0.01$, $Re = 4257$ (red dot in *(a)*).

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