TRIANGULAR CONSTELLATIONS IN FLOWS

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Abstract Particles advected on the surface of a fluid can exhibit fractal clustering. The local structure of a fractal set is described by its dimension D, which is the exponent of a power-law relating the mass $\mathcal{N}$ in a ball to its radius ε: $\mathcal{N} \sim \varepsilon^D$. It is desirable to characterise the *shapes* of constellations of points sampling a fractal measure, as well as their masses. The simplest example is the distribution of shapes of triangles formed by triplets of points, which we investigate for fractals generated by chaotic dynamical systems. The most significant parameter describing the triangle shape is the ratio z of its area to the radius of gyration squared. We show that the probability density of z has a phase transition: $P(z)$ is independent of ε and approximately uniform below a critical flow compressibility β_c, which we estimate. For $\beta > \beta_c$ the distribution appears to be described by two power laws: $P(z) \sim z^{\alpha_1}$ when $1 \gg z \gg z_c(\varepsilon)$, and $P(z) \sim z^{\alpha_2}$ when $z \ll z_c(\varepsilon)$.

INTRODUCTION

Fractal sets and measures play a pivotal role in many areas of physics [1, 2], including fluid dynamics, where it is known that particles advected by complex flows often exhibit fractal clustering [3]. Fractals which have nearly identical values of the dimension can have a very different appearance. It is desirable to develop means to characterise the shape of the internal structure of fractal distributions, because differences in the local structure of fractal sets may have important implications for properties such as light scattering [4] or network connectivity. Light scattering, for example, may be strongly enhanced in some directions by specular effects if scatterers tend to align on planes or lines. Recently, a 'spectal dimension' was defined which characterises anisotropy in the local structure of the fractal measures [5], but it is desirable to find simpler descriptions of the local shapes of fractal sets.

Here we address the simplest question about the internal shape-structure of a fractal set. Consider two randomly chosen particles in a ball of radius ε surrounding a reference point. Together with the test point, these define a triangle. The local structure can be described in greater detail by specifying the statistics of the shapes of these triangles.

We show that the distribution of triangle shapes is also associated with power-laws. It might be expected that the stretching action of the dynamics will exaggerate the prevalence of thin, acute-angled triangles. This expectation is only partially correct. We consider a one-parameter family of point fractals on the plane, and show that as the dimension is reduced below two, the prevalence of acute triangles remains constant until a critical dimension is reached. Below this critical dimension, the distribution of triangle shapes has a strong dependence upon dimension, and acute triangles become predominant.

NUMERICAL STUDIES OF ADVECTED TRIANGLES

As a concrete example of a dynamical process which generates a fractal measure we consider particles advected in a random flow in two dimensions [3]. This is the simplest case, but the techniques can be generalised to higher dimensions and more complex dynamical equations. The equation of motion is $\dot{\boldsymbol{r}} = \boldsymbol{u}(\boldsymbol{r}, t)$ where $\boldsymbol{u}(\boldsymbol{r}, t)$ is a random velocity field. In our numerical investigations we have used the map

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \boldsymbol{u}_n(\boldsymbol{x}_n)\sqrt{\delta t} \tag{1}$$

rather than a continuous flow. The velocity field $\boldsymbol{u}_n$ is chosen independently at each timestep, labelled by an integer n. It is constructed from two scalar fields, namely a stream function $\psi_n(x, y)$ and a scalar potential $\chi_n(x, y)$:

$$\boldsymbol{u}_n = \left(\frac{\partial \psi_n}{\partial y} + \beta \frac{\partial \chi_n}{\partial x}, -\frac{\partial \psi_n}{\partial x} + \beta \frac{\partial \chi_n}{\partial y}\right) . \tag{2}$$

Note that the flow is incompressible ($\mathrm{div}(\boldsymbol{u}_n) = 0$) when $\beta = 0$. For this reason β is termed the compressibility parameter of the flow. See [6] for full details of the model.

Our discussion of the triangle shapes will emphasise the coordinate z, defined as follows. Let $\mathcal{A}$ be the signed area of a triangle and $\mathcal{R}$ be the radius of gyration, then we write:

$$z = \frac{2\mathcal{A}}{\sqrt{3}\mathcal{R}^2} , \quad \mathcal{R}^2 = \frac{1}{3}\left[(\delta \boldsymbol{r}_1)^2 + (\delta \boldsymbol{r}_2)^2 + (\delta \boldsymbol{r}_1 - \delta \boldsymbol{r}_2)^2\right] \tag{3}$$

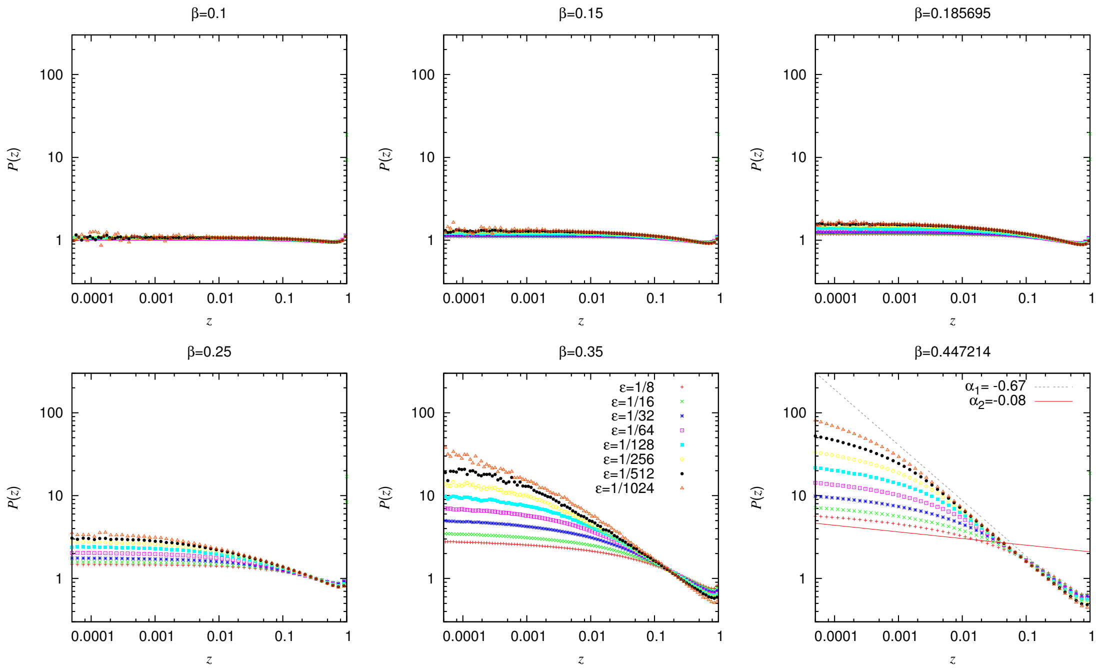

Figure 1. Probability density $P(z)$ for various compressibilities β including two special values: $\beta_c = 1/\sqrt{29} = 0.185\ldots$ is our estimate of the critical compressibility, and $\beta_1 = 1/\sqrt{5} = 0.447\ldots$ is the compressibility at which the correlation dimension satisfies $D_2 = 1$. Straight lines indicate estimates for α_1 and α_2 when $\beta = 1/\sqrt{5}$. Note that $P(z)$ is normalisable ($\alpha_1 > -1$) even at $\beta_1 = 1/\sqrt{5}$, where $D_2 = 1$

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where the $\delta \boldsymbol{r}_i$ are displacements of two points relative to the third, reference, point. Note that z may be negative, because $\mathcal{A}$ is defined via a vector product. The equilibrium distribution of diffusion on a spherical surface is a uniform probability density, corresponding to a uniform probability density for z. For a random scatter of points, Kendall [7] (see also [8]) showed that z has a uniform distribution on $[-1,1]$: $P(z) = \frac{1}{2}$. Note that thin, acute triangles correspond to small values of z. We concentrate upon the distribution $P(z)$ in the limit as $z \to 0$.

Figure 1 shows the numerically determined distribution of z for small triangular constellations formed by triplets of randomly chosen points inside a disc of radius $\varepsilon \ll \xi$. The plots show the probability distribution $P(z)$ for particles advected in six different random flows, with increasing values of the compressibility parameter β. In each case the probability distributions $P(z)$ are shown on double-logarithmic scales, for eight different values of ε. For small compressibility β, the distribution is approximately independent of the value of ε and uniform (apart from a cusp at $z = 1$ which arises because our sampling criterion is different from Kendall's, in that we require that the three points lie inside a disc of radius ε). When β exceeds a critical value β_c, $P(z)$ becomes dependent upon ε. It appears to be asymptotic to two power laws in the limit as $\varepsilon \to 0$: $P(z) \sim z^{\alpha_1}$ when z is small, but exceeds a value $z_c(\varepsilon)$ which decreases as $\varepsilon \to 0$, and $P(z) \sim z^{\alpha_2}$ for $z \ll z_c$.

ANALYSIS

The presentation, based upon a recently published paper [6], will describe an analytical theory which explains the surprising phase transition illustrated in figure 1. It discusses why $P(z)$ has power-law behaviour, why there is a critical compressibility β_c, and why the distribution $P(z)$ may have two exponents for $\beta > \beta_c$.

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