MARKOV CLOSURE FOR THE LUNDGREN-MONIN-NOVIKOV HIERARCHY OF VELOCITY INCREMENTS IN BURGERS TURBULENCE

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Abstract A central, yet unsolved issue in the longstanding problem of hydrodynamic turbulence is the closure problem of turbulence, which is due to the nonlinear character of the Navier-Stokes equation. We formulate the closure problem for the many-increment probability distributions (PDF's) in Burgers turbulence and introduce a new method for closing the hierarchy. To this end, we rely on the experimentally and numerically verified assumption in [1] that the turbulent cascade possesses a Markov property in scale down to the so-called Einstein-Markov length. The hierarchy is closed at the stage of the two-increment PDF corresponding to a three-point closure that allows for a description of intermittency effects, not captured by other closure approximations, i.e. Gaussian closures etc. The proposed closure also opens up a possible way to a perturbative treatment of the Navier-Stokes equation beyond the Einstein-Markov length in successively taking into account a larger and larger scale "history" of the system.

DESCRIPTION OF THE TURBULENT ENERGY CASCADE BY A MARKOV PROCESS IN SCALE

Despite of tremendous efforts to provide a statistical description of turbulence over the last century, a controlled perturbative treatment of the closure problem seems out of reach and we have to restrict ourselves to rather phenomenological considerations of the underlying processes. However, in our opinion, a very sophisticated phenomenology has been given by R. Friedrich and J. Peinke [1] in the last decade. They interpreted the turbulent energy cascade as a stochastic process in scale that possesses Markov properties, which could be verified in a free jet experiment. The central notion of the Markov approach is the n-increment PDF $f_n(v_n, r_n; ...; v_1, r_1)$ of the longitudinal velocity increments $v(\mathbf{r}, \mathbf{x}, t) = (\mathbf{u}(\mathbf{x}+\mathbf{r}, t) - \mathbf{u}(\mathbf{x}, t)) \cdot \mathbf{e}_r$. It is known from the theory of stochastic processes that we are able to express the n-increment PDF as a product of the $n-1$-increment PDF and a conditional probability determined by

$$p(v_n, r_n|v_{n-1}, r_{n-1}; ...; v_1, r_1) = \frac{f_n(v_n, r_n; ...; v_1, r_1)}{f_{n-1}(v_{n-1}, r_{n-1}; ...; v_1, r_1)} \tag{1}$$

Now, the central interpretation of the energy cascade in this approach is that a structure on scale $n-1$ becomes unstable and decays into smaller structures on scale n. The localness of these interactions can thus be interpreted as a Markov property of the conditional probabilities, namely

$$p(v_n, r_n|v_{n-1}, r_{n-1}; ...; v_1, r_1) = p(v_n, r_n|v_{n-1}, r_{n-1}) \tag{2}$$

meaning a tremendous reduction of our problem: The entire statistics of the turbulent cascade is now determined by these transition probabilities and especially their evolution in scale r_n. This can be seen from the n-increment PDF, which can be rewritten as a chain of transition probabilities

$$f_n(v_n, r_n; ...; v_1, r_1) = p(v_n, r_n|v_{n-1}, r_{n-1})p(v_{n-1}, r_{n-1}|v_{n-2}, r_{n-2})...p(v_2, r_2|v_1, r_1)f(v_1, r_1) \tag{3}$$

Furthermore, Friedrich and Peinke deduced that the one-increment PDF as well as the transition probabilities obey a Fokker-Planck equation in scale of the form

$$-\frac{\partial}{\partial r_i} f(v_i, r_i) = \left[-\frac{\partial}{\partial v_i} D^{(1)}(v_i, r_i) + \frac{\partial^2}{\partial v_i^2} D^{(2)}(v_i, r_i)\right] f(v_i, r_i) \tag{4}$$

with the drift- and diffusion coefficients $D^{(1)}(v_i, r_i) = -\frac{3+\mu}{9r_i} v_i$ and $D^{(2)}(v_i, r_i) = \frac{\mu}{18r_i} v_i^2$ corresponding to the K62 log-normal phenomenology. Here, the intermittency factor μ occurs as a free parameter, which has to be determined from the experiment. Furthermore, in subsequent works [2, 3], it became clear that the Markov property breaks down at small scales, at the so-called Einstein-Markov length that is of the order of the Taylor length. In addition, since this is a purely phenomenological way of deriving the PDF's, so far no connection to the basic fluid dynamical equations, i.e. the Navier-Stokes equation could be made. In this context, a promising candidate of obtaining such a connection seems to be the Lundgren-Monin-Novikov hierarchy [4, 5, 6] that is formally similar to the BBGKY hierarchy of statistical physics as it is discussed in the following section.

THE LUNDGREN-MONIN-NOVIKOV HIERARCHY OF VELOCITY INCREMENTS IN BURGERS TURBULENCE

We consider the Burgers equation

$$\frac{\partial}{\partial t} u(x, t) + u(x, t)\frac{\partial}{\partial x} u(x, t) = \nu \frac{\partial^2}{\partial x^2} u(x, t) + F(x, t) \tag{5}$$

with forcing $F(x,t)$ that is white-noise in time $\langle F(x,t)F(x',t)\rangle = \chi(x-x')\delta(t-t')$, where $\chi(x-x')$ is the spatial correlation function of the forcing procedure. Our goal is to derive an evolution equation for the one-increment PDF $f(v_1,R_1,r_1,t) = \langle \delta(v_1 - u(R_1+r_1,t) + u(R_1,t))\rangle$, in making contact to the Burgers equation (5), in analogy to the works [4, 7] for Navier-Stokes. We obtain

$$\begin{aligned} & \frac{\partial}{\partial t} f_1(v_1,r_1,t) + v_1 \frac{\partial}{\partial r_1} f_1(v_1,r_1,t) \\ = \; & -2\int_{-\infty}^{v_1} \mathrm{d}v \frac{\partial}{\partial r_1} f_1(v,r_1,t) - 3\nu \frac{\partial}{\partial v_1} \lim_{r_2 \to r_1} \frac{\partial^2}{\partial r_2^2} \int \mathrm{d}v_2 v_2 f_2(v_2,r_2;v_1,r_1,t) + [\chi(0)-\chi(r_1)] \frac{\partial^2}{\partial v_1^2} f_1(v_1,r_1,t) \end{aligned} \tag{6}$$

which is an unclosed equation due to the dissipative term, which couples to the two-increment PDF and is referred to as dissipation anomaly, since it is non-vanishing even for the case as $\nu \to 0$, due to the presence of shocks in the velocity field. Under the assumption of stationarity, Eq. (6) reduces to an evolution equation in scale for the one-increment PDF that shows similarities to the Fokker-Planck equation (4) discussed in the preceding section. A similar equation governs the evolution of the two-increment equation in scale, namely

$$\begin{aligned} & v_1 \frac{\partial}{\partial r_1} f_2(v_2,r_2;v_1,r_1) + v_2 \frac{\partial}{\partial r_2} f_2(v_2,r_2;v_1,r_1) \\ = \; & -\int_{-\infty}^{v_1} \mathrm{d}v \frac{\partial}{\partial r_1} f_2(v_2,r_2;v,r_1) - \int_{-\infty}^{v_2} \mathrm{d}v \frac{\partial}{\partial r_2} f_2(v,r_2;v_1,r_1) + \lim_{r_3 \to 0} \frac{\partial}{\partial r_3} \int \mathrm{d}v_3 v_3 f_3(v_3,r_3;v_2,r_2;v_1,r_1) \\ & -3\nu \left[\frac{\partial}{\partial v_1} \lim_{r_3 \to r_1} + \frac{\partial}{\partial v_2} \lim_{r_3 \to r_2} \right] \frac{\partial^2}{\partial r_3^2} \int \mathrm{d}v_3 v_3 f_3(v_3,r_3;v_2,r_2;v_1,r_1) \\ & + \left[[\chi(0)-\chi(r_1)] \frac{\partial^2}{\partial v_1^2} + [\chi(0)-\chi(r_1)-\chi(r_2)+\chi(r_1-r_2)] \frac{\partial^2}{\partial v_1 \partial v_2} + [\chi(0)-\chi(r_2)] \frac{\partial^2}{\partial v_2^2} \right] f_2(v_2,r_2;v_1,r_1) \end{aligned} \tag{7}$$

We rewrite the three-increment PDF by the use of conditional probabilities according to

$$\begin{aligned} & f_3(v_3,r_3;v_2,r_2;v_1,r_1) \\ = \; & p(v_3,r_3|v_2,r_2;v_1,r_1) f_2(v_2,r_2;v_1,r_1) \end{aligned} \tag{8}$$

In order to close the equation at this point, we assume that the Markov property (2) holds in a first approximation. In this case we obtain a closed system of equations (6) and (7) for the one-increment pdf $f(v_1,r_1)$ and the transition probability $p(v_2,r_2|v_1,r_1)$. Obviously, this approximation has to be examined carefully, since in the limit procedures in (7), we approach inevitably the Einstein-Markov length, where Eq. (2) is violated. This can be done in comparing our approximate results to results obtained from direct numerical simulations of Eq. (5) like, for instance, the typical evolution in scale of the one-increment PDF depicted in Fig.1. Here, the Markov closure should be a good approximation for the right part of the PDF that corresponds to linear ramps in between shocks, whereas the reconstruction of the left part might overexert our closure assumption. However, our belief is that by taking into account further memory effects in Eq. (2), we should be able to provide a systematic perturbation theory for the Lundgren-Monin-Novikov hierarchy beyond the Einstein-Markov length.

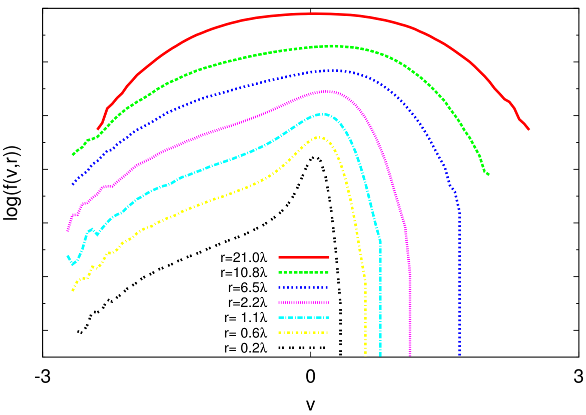

Figure 1. Evolution of the one-increment PDF in scale obtained from direct numerical simulations (resolution $N = 8192$, $\mathrm{d}x = 0.0049$, viscosity $\nu = 0.03$, spatial correlation of forcing $\sim k^{-1}$ in Fourier space, Kolmogorov length $\eta = 0.112$, Taylor length $\lambda = 0.225$): On large scales the one-increment PDF is close to Gaussian whereas it develops a pronounced asymmetry as it tends to smaller scales due to the presence of shocks in the left tail. Finally, it approaches the gradient PDF that is amenable to instanton calculations of the right and left tail.

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