HIGH RESOLUTION SIMULATIONS OF RANDOM FIELDS AND IMPLICATIONS ON STOCHASTIC MODELLING OF TURBULENCE

Rodrigo Miranda Pereira[1,2] & Laurent Chevillard[1]
[1]Laboratoire de Physique de l'École Normale Supérieure de Lyon, 46 allée d'Italie F-69007 Lyon, France
[2]CAPES Foundation, Ministry of Education of Brazil, Brasília – DF, 70040-020

Abstract We study through high resolution numerical simulations unknown features of a random velocity field based on the so-called multiplicative chaos, previously proposed in the literature to model turbulence [3]. We investigate the influence of the intermittency parameter, observing the behaviour of the field as it is changed. In this way, we can determine whether corrections to the 4/5 law exist and pursue some properties that have not been analytically computed. We also study the effects on other realistic properties such as the preferential alignment of vorticity.

THE RANDOM VELOCITY FIELD

The development of synthetic velocity fields to model turbulence has received a lot of attention in recent years. On the one hand there are important practical applications, for instance to save substantial computation time producing inputs for numerical simulations [1], but one is also interested in creating tractable mathematical objects reproducing the archetypal properties of turbulence in order to gain some theoretical insight on the mechanisms behind the Navier-Stokes equations and the behaviour of its turbulent solutions.

In this picture, Robert and Vargas [2] proposed a family of stochastic homogeneous and isotropic fields based on the so-called multiplicative chaos, a process generated from the exponential of another stochastic process. It explicitly exhibited Kolmogorov's 4/5 law and intermittency, however it was not incompressible. Incompressibility could be achieved combining the components in a Biot-Savart fashion but at the cost of getting symmetrical velocity increments, ruining one of the most important features of turbulence. This issue was only solved with a structural change, the generalisation to a matrix-valued multiplicative chaos which finds an interesting analogy with the dynamics of vorticity stretching [3]. The idea is to take the exponential of the homogeneous and isotropic matrix field

$$X^{\epsilon}(\boldsymbol{y}) = \sqrt{\frac{15}{32\pi}} \int_{|\boldsymbol{y}-\boldsymbol{z}|\leq L} \left\{ \frac{(\boldsymbol{y}-\boldsymbol{z}) \otimes [(\boldsymbol{y}-\boldsymbol{z}) \wedge \mathrm{d}\boldsymbol{W}(\boldsymbol{z})]}{|\boldsymbol{y}-\boldsymbol{z}|_{\epsilon}^{7/2}} + \frac{[(\boldsymbol{y}-\boldsymbol{z}) \wedge \mathrm{d}\boldsymbol{W}(\boldsymbol{z})] \otimes (\boldsymbol{y}-\boldsymbol{z})}{|\boldsymbol{y}-\boldsymbol{z}|_{\epsilon}^{7/2}} \right\}, \quad (1)$$

whose components are log-correlated over the integral length scale L, to create the following vector velocity field

$$\boldsymbol{u}^{\epsilon}(\boldsymbol{x}) = \int \varphi_L(\boldsymbol{x}-\boldsymbol{y}) \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|_{\epsilon}^{13/6}} \wedge \mathrm{e}^{\gamma X^{\epsilon}(\boldsymbol{y})} \mathrm{d}\boldsymbol{W}(\boldsymbol{y}). \quad (2)$$

In (1), $\otimes$ represents the tensor product $(\boldsymbol{x} \otimes \boldsymbol{y})_{ij} \equiv x_i y_j$. $\mathrm{d}\boldsymbol{W}$ is a three-dimensional vector white noise and the same one is used both in (1) and (2), since correlation between them is crucial to reproduce realistic results. The large scale cut-off function φ_L ensures that the integral converges at infinity, rapidly decaying for $|\boldsymbol{x}| > L$, and the small scale cut-off ϵ regularizes the function $1/|\boldsymbol{x}|$, being interpreted as the Kolmogorov dissipative scale. c_ϵ is a normalization constant added so the variance is finite in the limit $\epsilon \to 0$. The intermittency parameter γ is central: one has a gaussian field when it is zero and intermittency develops as it increases. Its value was fixed as $\gamma \approx 0.26$ through the fitting of the flatness curve in relation to experimental and DNS data. It is worth to mention that the numerical factor $\sqrt{15/32\pi}$ is chosen so in the limit $\epsilon \to 0$ one has $\langle X_{11}^{\epsilon}(\boldsymbol{x}) X_{11}^{\epsilon}(0) \rangle \sim \ln(L/|x|)$ as $|x| \to 0$, with no additional numerical factors.

STATISTICS

Numerical simulations showed that this incompressible field indeed exhibited skewed velocity increments and, furthermore, reproduced other typical turbulence phenomena such as vorticity alignments and the asymmetry of the RQ-Plane. Nevertheless, the complexity brought by the matrix multiplicative chaos prevents the derivation of analytical results. It has not been possible to explicitly calculate the structure functions, so one may not know, for example, if there are small intermittent corrections depending on γ to the third order structure function (and consequently deviations from the 4/5 law) or how these corrections develop for other structure functions. Even the process' variance is unknown, so one is not able to define the proper normalization constant which gives a finite variance in the limit $\epsilon \to 0$.

This work intends to investigate these issues and the influence of γ through high-resolution parallel simulations and also discuss simplified cases where analytical results can be obtained, giving some insight on what to expect from the statistical properties of (2). Figure 1 depicts for instance the third order structure functions calculated from simulations on a cube with 2048^3 points for different intermittency parameters. It shows the increments are skewed in reasonable agreement with the Kolmogorov scaling (thus the fields incorporate energy transfers) and moreover that the multiplicative constant depends on γ. Evidence of corrections to the 4/5 law are found when comparing the unsigned third order structure

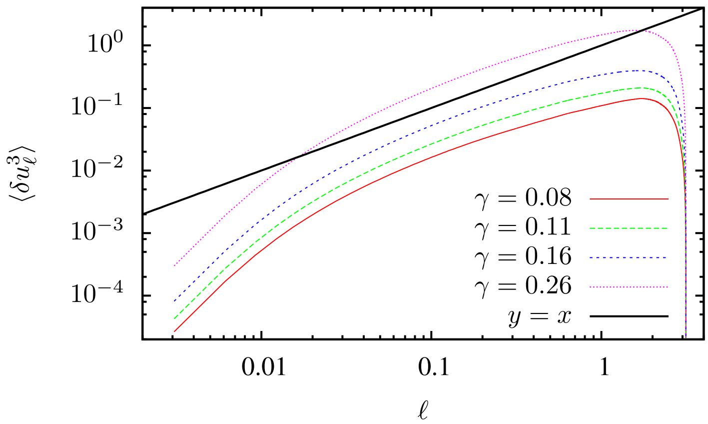
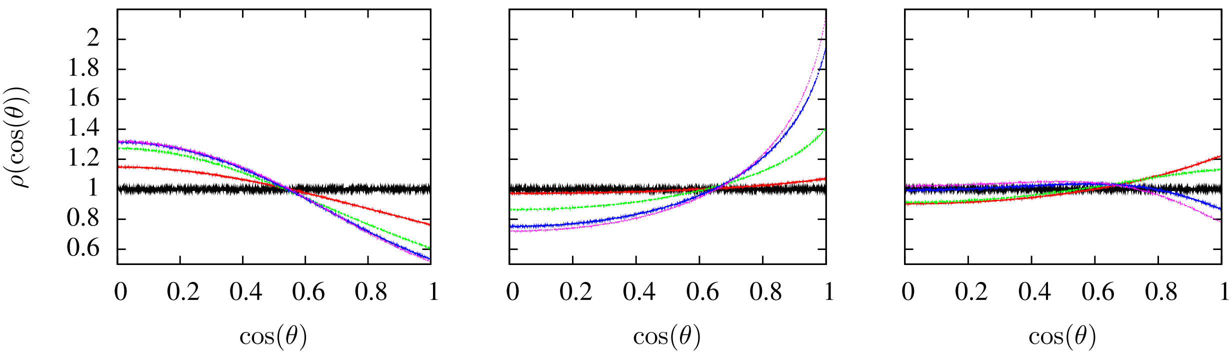


Figure 1. Third order structure function.

functions to the gaussian case, providing a way to modify (2) to ensure an asymptotic Kolmogorov scaling. We also analyze the proper normalization constant, checking how it changes with the small scale cut-off ϵ as well as with γ. The effect of the intermittency parameter on other more intricate properties of turbulence is considered as well, such as the preferential alignment of the vorticity vector. In figure 2 we plot the probability distribution functions (PDFs) of the cosine of the angles between the vorticity and the directions defined by the eigenvectors associated with the lowest, the intermediate and the highest eigenvalues of the rate-of-strain tensor, for different values of γ. One sees that for the gaussian case $\gamma = 0$ there is no preferential alignment, while as γ increases a substantial alignment with the direction associated to the intermediate eigenvalue is observed, along with a tendency to remain normal to the lowest eigenvalue direction. Variations with respect to the highest eigenvalue direction are less sensible. It is interesting to note that there seems to be a saturation of the alignments as γ increases too much, approaching the limiting value $1/\sqrt{6} \approx 0.41$ for which (2) is defined.



(a) Lowest (b) Intermediate (c) Highest

Figure 2. PDFs of the cosine of the angles between the vorticity vector and the eigenvectors of the rate-of-strain tensor. Red: $\gamma = 0.08$. Green: $\gamma = 0.18$. Blue: $\gamma = 0.26$. Magenta: $\gamma = 0.34$. Black: gaussian case $\gamma = 0$.

References

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