UNIVERSAL STATISTICS OF POINT VORTEX TURBULENCE: THE DOUBLY-PERIODIC DOMAIN

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Abstract A new solution technique is used to obtain a statistical description of the motion of N point vortices, evolving in the usual two-dimensional doubly-periodic domain, in the limit $N \to \infty$. In contrast to previous approaches such as [5], a mean-field approximation is not used, meaning that the theory can describe the full (divergent) stationary energy spectrum associated with the vortex motion. An explicit formula for this energy spectrum is obtained, which is compared with direct numerical simulations with $N = 50$ vortices, and excellent agreement is found across a range of vortex interaction energies (Hamiltonian of the point vortex system). The implications for understanding related non-equilibrium systems such as 2D classical turbulence and superfluid turbulence are discussed.

BACKGROUND

The point vortex model has long [9, 5, 3, 1] been considered an instructive, albeit idealised, model for both two-dimensional fluid turbulence and of ‘guiding-center’ plasmas. Interest has been revived recently by the prospect of experimental realisations of two-dimensional quantum turbulence in superfluids, because in the relevant parameter regime the point vortex model can be an excellent description of the dynamics as modelled by the Gross-Pitaevskii equation [2, 12].
The most widely known analysis of the point vortex system, following [5], uses mean-field statistical mechanics. The outcome is a mean-field equation (e.g. the sinh-Poisson equation) which describes the streamfunction-vorticity relationship of a steady (or steadily propagating) time-mean circulation. A starting point of mean-field theory is that all of the energy of the flow is contained in this time-mean flow (or condensate). As a result, the mean-field theory and its generalisations [11] have had successes [8] in predicting the late-time states of 2D turbulent flows (i.e. following condensation).
The mean-field approach, however, gives no information about the energy spectrum of 2D turbulence at earlier times. In contrast the statistical theory of Kraichnan [6, 7], using a spectral truncation of the Euler equations, makes predictions for the equilibrium spectrum under the assumptions that total energy and enstrophy are conserved. An open question is whether point vortex predictions of the same type are possible. As a step in this direction [4] showed that results related to energy partitioning could be obtained using the cumulant expansion method of [10]. The subject of the presentation is a new method recently discovered, allowing equilibrium energy spectra to be calculated explicitly, and which can be applied to the doubly periodic domain.

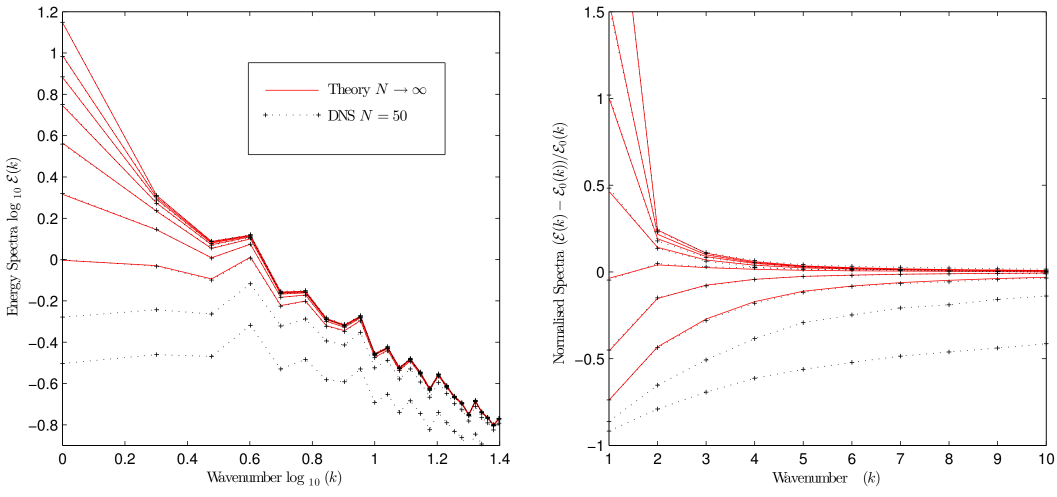

Figure 1: Left: Log energy spectrum $\log_{10} \mathcal{E}(k)$ versus log wavenumber $\log_{10}(k)$ for nine levels of the vortex interaction energy $\varepsilon = \{-5, -3, -2, -1, 0, 1, 2, 3, 5\}\ \sigma_\varepsilon$. Theoretical results - red curves, DNS black dotted curves. Right: The same data, showing the normalised ‘excess’ spectra at low wavenumbers.

RESULTS

The point vortex model is a Hamiltonian system and the statistics of long-time integrations are determined by the microcanonical ensemble, which is defined by all microstates with Hamiltonian $H = \varepsilon$, the constant vortex interaction energy. Defining $E_{\boldsymbol{k}}$ to be the energy in the mode with wavenumber $\boldsymbol{k}$, in the doubly-periodic domain it is possible to write [7],

$$H = \varepsilon = \sum_{\boldsymbol{k}} E_{\boldsymbol{k}} - E_{\boldsymbol{k}}^0 \qquad (1)$$

where $E^0_{\boldsymbol{k}} \propto k^{-2}$ is the expected energy had the same vortices been distributed randomly. Notice that the summation in (1) applied to either $E_{\boldsymbol{k}}$ or $E^0_{\boldsymbol{k}}$ individually is divergent, since (as is well-known) all point vortex energy spectra $\sim k^{-1}$ at high wavenumbers.

The programme followed to obtain the microcanonical statistics, which holds for any distribution of vortex circulations as $N \to \infty$, is:

1. The uniform statistics, generated when the vortices are distributed at random in the domain, are considered first. The central limit theorem is used to obtain the (normally-distributed) statistics of $\omega_{\boldsymbol{k}}$, the Fourier coefficient of the vorticity field for wavenumber $\boldsymbol{k}$

2. It follows that, under the uniform statistics $E_{\boldsymbol{k}}$ has an exponentially distributed pdf, with parameter k^{-2}.

3. Exact results for summation of exponential random variables are then used to sum (1) to obtain an explicit formula for the density of states function $W(\varepsilon)$ (pdf of H under the uniform statistics).

4. Denoting pdfs under the microcanonical ensemble by $p_\varepsilon(\cdot)$ and pdfs under the uniform statistics by $p_0(\cdot)$, Bayes' theorem can be used to get the microcanonical pdf of $E_{\boldsymbol{k}}$ as follows

$$p_\varepsilon(e_k) = \frac{W_{\boldsymbol{k}}(\varepsilon - e_k)p_0(e_k)}{W(\varepsilon)}, \tag{2}$$

   where $W_{\boldsymbol{k}}(\varepsilon)$ denotes the density of states when the wavevector $\boldsymbol{k}$ is omitted from the calculation.

5. The mean of of the pdf $p_\varepsilon(e_k)$ gives $\langle E_{\boldsymbol{k}} \rangle$ and the energy spectrum $\mathcal{E}(k;\varepsilon)$ follows from summation over wavenumber shells.

In Fig. 1 the energy spectra thus obtained are compared with results from DNS of the point vortex equations obtained following the method of [13]. $N = 50$ vortices are used with unit positive and negative circulations. Long integrations ($\approx 10^5$ circulation times) are performed with $\varepsilon = -5\sigma_\varepsilon$ to $+5\sigma_\varepsilon$, where σ_ε is the standard deviation of $W(\varepsilon)$. The results show excellent agreement between theory and DNS. Notably, once $\varepsilon > 0$ all further energy accumulates in the gravest mode of the system (the condensate) whereas for $\varepsilon < 0$ the energy deficit (relative to $\mathcal{E}^0(k)$) is more evenly distributed in wavenumber space.

OUTLOOK AND CONCLUSIONS

The theory described above, apart from accurately describing the point vortex statistics, is a interesting starting point for re-examination of non-equilibrium systems such as decaying 2D turbulence and 2D quantum turbulence, following [3, 1]. The point vortex equilibrium spectra will be accurate only if various conditions are met: the domain area fraction of vortices must be small, the time-scale for adjustment to equilibrium must be faster than the time-scale of non-equilibrium processes (e.g. vortex merger), and the energy must not be so large as to enter the sinh-Poisson regime. A rigorous numerical examination these conditions will be the goal of future work.

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