# KRAICHNAN-LEITH-BATCHELOR SIMILARITY THEORY AND TWO-DIMENSIONAL INVERSE CASCADES

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*Abstract* We study the scaling properties and Kraichnan-Leith-Batchelor (KLB) theory of forced inverse cascades in generalized two-dimensional (2D) fluids ($\alpha$-turbulence models) simulated at resolution $8192^2$. We consider $\alpha = 1$ (surface quasigeostrophic flow), $\alpha = 2$ (2D vorticity dynamics) and $\alpha = 3$. The forcing scale is well-resolved, a direct cascade is present and there is no large-scale dissipation. Coherent vortices spanning a range of sizes, most larger than the forcing scale, are present for both $\alpha = 1$ and $\alpha = 2$. The active scalar field for $\alpha = 3$ contains comparatively few and small vortices. The energy spectral slopes in the inverse cascade are steeper than the KLB prediction $-(7-\alpha)/3$ in all three systems. Since we stop the simulations well before the cascades have reached the domain scale, vortex formation and spectral steepening are not due to condensation effects; nor are they caused by large-scale dissipation, which is absent. One- and two-point pdfs, hyperflatness factors and structure functions indicate that the inverse cascades are intermittent and non-Gaussian over much of the inertial range for $\alpha = 1$ and $\alpha = 2$, while the $\alpha = 3$ inverse cascade is much closer to Gaussian and non-intermittent. For $\alpha = 3$ the steep spectrum is close to that associated with enstrophy equipartition. Continuous wavelet analysis shows approximate KLB scaling $\mathcal{E}(k) \propto k^{-2}$ ($\alpha = 1$) and $\mathcal{E}(k) \propto k^{-5/3}$ ($\alpha = 2$) in the interstitial regions between the coherent vortices. Our results demonstrate that coherent vortex formation ($\alpha = 1$ and $\alpha = 2$) and non-realizability ($\alpha = 3$) cause 2D inverse cascades to deviate from the KLB predictions, but that the flow between the vortices exhibits KLB scaling and non-intermittent statistics for $\alpha = 1$ and $\alpha = 2$. The results will appear in Burgess *et al.* (2015), which has been accepted to the *Journal of Fluid Mechanics*.

## BACKGROUND AND MAIN RESULTS

The extent to which Kraichnan-Leith-Batchelor (KLB) similarity theory (Kraichnan , 1967; Leith , 1968; Batchelor , 1969) describes inverse cascades is a major unresolved issue in 2D turbulence. Using a pseudospectral code at resolution $8192^2$ we simulate inverse cascades in generalized 2D fluids, also known as $\alpha$ turbulence models (Pierrehumbert *et al.* , 1994). In these models an active scalar $\theta$ is advected by a velocity field to which it is functionally related, $\theta = (-\Delta)^{\alpha/2}\psi$, where $\psi$ is the streamfunction and $\alpha$ is a parameter controlling the scale separation between $\psi$ and $\theta$. When $\alpha = 2$, the active scalar is the familiar vorticity $\omega = -\nabla^2\psi$ and the unforced inviscid system reduces to 2D Euler flow. Strongly rotating quasigeostrophic flows and plasmas in strong magnetic fields both correspond to $\alpha = -2$, while surface quasi-geostrophic dynamics (SQG) corresponds to $\alpha = 1$.

Similarity theory, which assumes scale-invariant inertial ranges in which transfers are spectrally local, predicts that the generalized energy spectrum $\mathcal{E}(k)$ follows a power law $\mathcal{E}(k) \propto k^{-(7-\alpha)/3}$ in the inverse cascade. For $\alpha = 2$ this yields the well-known $-5/3$ law for the kinetic energy (KE) spectrum. Despite the archetypal status of the inverse KE cascade for $\alpha = 2$, there is disagreement about its phenomenology and statistical characteristics. Some authors, e.g. Boffetta & Ecke (2012), have found that this cascade is well-described by self-similar inertial range theory, lacks coherent vortices, has almost Gaussian statistics, and is non-intermittent. Other studies, e.g. Vallgren (2011), have found spectra significantly steeper than $k^{-5/3}$ in the inverse KE cascade, suggesting the KLB scaling is not generic.

Most of the studies that found $k^{-5/3}$ scaling in the inverse KE cascade used forcing near the largest resolved wavenumber, which does not allow an enstrophy cascade to develop, and/or a large-scale drag or hypoviscosity, which can both disrupt vortex formation and cause condensation-like effects. In our simulations the forcing scale is well-resolved, an enstrophy cascade is present, and there is no large-scale dissipation, so spectral steepening is not due to condensation-like effects. We find that both the $\alpha = 1$ and $\alpha = 2$ inverse cascades are populated by vortices with a range of sizes, accompanied by steep spectra (figure 1), non-Gaussian and intermittent statistics. We use continuous wavelet analysis to study the field between the vortices, and find that it exhibits approximate KLB scaling (figure 2). In contrast to the other two systems, $\alpha = 3$ contains no large or persistent vortices, but the spectrum is still steeper than the KLB scaling, as predicted by Burgess & Shepherd (2013).

To what extent KLB similarity theory describes 2D inverse cascades appears to be a rather complex question. Sensitivity to simulation parameters, the various tendencies of 2D fluids to form coherent structures, and the realizability of the KLB similarity solutions as inverse cascades all impact the answer. Though coherent structures form when flows with $\alpha = 1$

and $\alpha = 2$ are forced at resolved scales, causing spectral steepening, non-Gaussianity, and intermittency, KLB theory remains a good description of the interstitial fields in these systems.

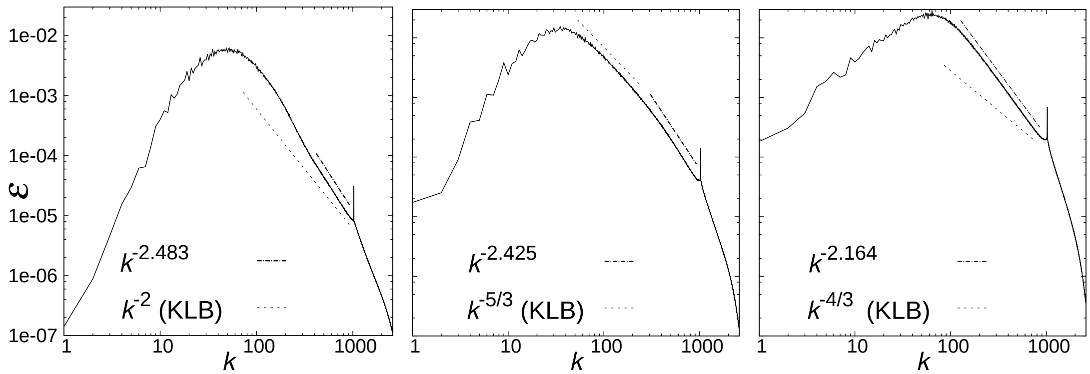


**Figure 1.** Inverse cascade energy spectra for $\alpha = 1$, $\alpha = 2$, and $\alpha = 3$, with KLB scaling for comparison.

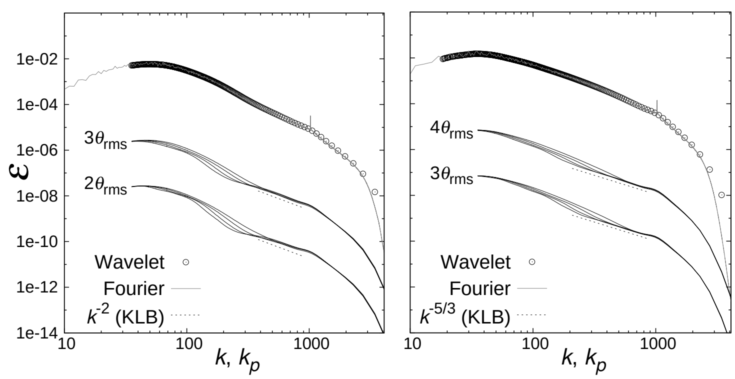


**Figure 2.** Wavelet and Fourier spectra for $\alpha = 1$ (left) and $\alpha = 2$ (right). The Fourier spectrum (gray) is overlain on the globally averaged wavelet spectrogram (black circles). Wavelet spectrograms for the interstitial flow (solid black lines) also appear together with KLB scaling (dashed black lines).

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