HAMILTONIAN FORMALISM FOR WEAK TURBULENCE OF INERTIAL WAVES IN ROTATING FLUIDS

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Abstract We present the Hamiltonian description of incompressible rotating fluid by using the Clebsch variables. We find the transformation which allows us to present the three-wave interaction Hamiltonian in normal variables in simple explicit form. We analyze the three-wave interaction amplitude and find new anisotropic spectra. Finally we study the convergence of integrals and present the kinetic equation.

INTRODUCTION

The inertial waves in incompressible rotating fluid obey the following dispersion law:

$$\omega = 2\Omega|\cos\theta|, \tag{1}$$

where Ω is the value of the angular velocity, θ - is the angle between the wave vector $\boldsymbol{k}$ and the axes of rotation. When the rotation is rapid the waves interact weakly, that means that we deal with so-called "weak turbulence". The problem attracts the attention of experimenters [3] and its theoretical study is far from complete. The theory of weak turbulence was studied using a helicity decomposition in [2]. Meanwhile, the Hamiltonian formalism has not been developed.

HAMILTONIAN DESCRIPTION

We construct the Hamiltonian formalism by using the well-known fact that the Euler equation for an incompressible fluid

$$\partial_t \boldsymbol{v}(\boldsymbol{r},t) = -(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{\nabla}p\,, \tag{2a}$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{v}(\boldsymbol{r},t) = 0 \tag{2b}$$

can be written as a Hamiltonian system, using Clebsch representation for the velocity field [1]:

$$\boldsymbol{v}(\boldsymbol{r},t) = \lambda(\boldsymbol{r},t)\boldsymbol{\nabla}\mu(\boldsymbol{r},t) + \boldsymbol{\nabla}\Phi(\boldsymbol{r},t), \tag{3}$$

where $\Phi(\boldsymbol{r},t)$ is the potential which is uniquely determined from the proper boundary conditions and the incompressibility condition (2b). $\lambda(\boldsymbol{r},t)$ and $\mu(\boldsymbol{r},t)$ are independent canonically-conjugated fields (Clebsch variables). In the rotating reference frame the velocity field can be written in the following form:

$$\boldsymbol{v}(\boldsymbol{r},t) = \sqrt{2\Omega}(\lambda(\boldsymbol{r},t)\mathbf{e}_y - \mu(\boldsymbol{r},t)\mathbf{e}_x) + \lambda(\boldsymbol{r},t)\boldsymbol{\nabla}\mu(\boldsymbol{r},t) + \boldsymbol{\nabla}\Phi(\boldsymbol{r},t)\;. \tag{4}$$

Let us use the notation $\boldsymbol{v}(\boldsymbol{r},t) = \boldsymbol{v}_0(\boldsymbol{r},t) + \boldsymbol{v}_1(\boldsymbol{r},t)$, where $\boldsymbol{v}_0(\boldsymbol{r},t) = \sqrt{2\Omega}(\lambda\mathbf{e}_y - \mu\mathbf{e}_x)$ and $\boldsymbol{v}_1(\boldsymbol{r},t) = \lambda(\boldsymbol{r},t)\boldsymbol{\nabla}\mu(\boldsymbol{r},t) + \boldsymbol{\nabla}\Phi(\boldsymbol{r},t)$. The dispersion law (1) allows the three-wave interaction processes. Thus the Hamiltonian has quadratic part, three-wave and four-wave interaction parts: $\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4$. Here

$$\mathcal{H}_2 = \frac{1}{2}\int|\boldsymbol{v}_0(\boldsymbol{k})|^2 d\boldsymbol{k},\;\; \mathcal{H}_3 = \frac{1}{2}\int\left[\boldsymbol{v}_0(\boldsymbol{k})\cdot\boldsymbol{v}_1^*(\boldsymbol{k}) + \boldsymbol{v}_0^*(\boldsymbol{k})\cdot\boldsymbol{v}_1(\boldsymbol{k})\right]d\boldsymbol{k},\;\; \mathcal{H}_4 = \int\boldsymbol{v}_1(\boldsymbol{k})\cdot\boldsymbol{v}_1^*(\boldsymbol{k})d\boldsymbol{k}\,, \tag{5}$$

where $\boldsymbol{v}_0(\boldsymbol{k})$ and $\boldsymbol{v}_1(\boldsymbol{k})$ are the Fourier transform of $\boldsymbol{v}_0(\boldsymbol{r})$ and $\boldsymbol{v}_1(\boldsymbol{r})$. We write (5) in the Clebsch variables and the apply the canonical transformation which allows us to present the three-wave interaction Hamiltonian in simple explicit form. The dispersion law (1) and the conservation law for the three-wave interaction process:

$$\omega_{\boldsymbol{k}_1+\boldsymbol{k}_2} = \omega_{\boldsymbol{k}_1} + \omega_{\boldsymbol{k}_2}, \tag{6}$$

lead to the small-scale instability. We study this process by analyzing the three-wave interaction amplitude. We find new anisotropic spectra, study the convergence of integrals and present the kinetic equation.

References

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