ROLE OF THE STRAIN-RATE TENSOR IN TURBULENT SCALAR-TRANSPORT MODELING

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Abstract We examine the geometric orientation of the subfilter-scale scalar-flux vector in homogeneous isotropic turbulence. Vector orientation is determined using the eigenframe of the resolved strain-rate tensor. The Schmidt number is kept sufficiently large so as to leave the velocity field, and hence, the strain-rate tensor, unaltered by filtering in the viscous-convective subrange. Strong preferential alignment is observed for the case of Gaussian and box filters, whereas the sharp-spectral filter leads to close to a random orientation. The orientation angle obtained with the Gaussian and box filters is largely independent of the filter-width and the Schmidt number. It is shown that the alignment direction observed numerically using these two filters is predicted very well by the tensor-diffusivity model. Further *a-priori* tests indicate poor alignment of the Smagorinsky and stretched vortex model predictions with the exact subfilter flux.

INTRODUCTION

Large Eddy Simulations (LES) reduce the computational cost of turbulent flows to manageable levels by resolving the larger length scales to a certain extent, and employing models to represent the dynamics of the smaller, unresolved scales. The mathematical term to be modeled in LES of scalar transport, referred to as the subfilter-scale scalar-flux (SFF), is obtained by applying a homogeneous spatial filtering operation $(\widetilde{\cdot})$ to the convection-diffusion equation:

$$\begin{aligned} \frac{\partial \widetilde{\phi}}{\partial t} + \nabla \cdot \left(\widetilde{\boldsymbol{u}} \widetilde{\phi} \right) &= \mathcal{D} \nabla^2 \widetilde{\phi} - \nabla \cdot \boldsymbol{\tau}_\phi & (1) \\ \boldsymbol{\tau}_\phi &= \widetilde{\boldsymbol{u}\phi} - \widetilde{\boldsymbol{u}} \widetilde{\phi} & (2) \end{aligned}$$

A variety of models are available in the literature for emulating $\boldsymbol{\tau}_\phi$, namely, variations of the Smagorinsky model, the stretched vortex model, the similarity model, and the tensor-diffusivity model [1]. One of the important properties of these models that must be tested is the predicted orientation of the $\boldsymbol{\tau}_\phi$ vector. Accurate knowledge of this orientation is crucial for correctly determining the amount of subfilter scalar variance dissipation given by:

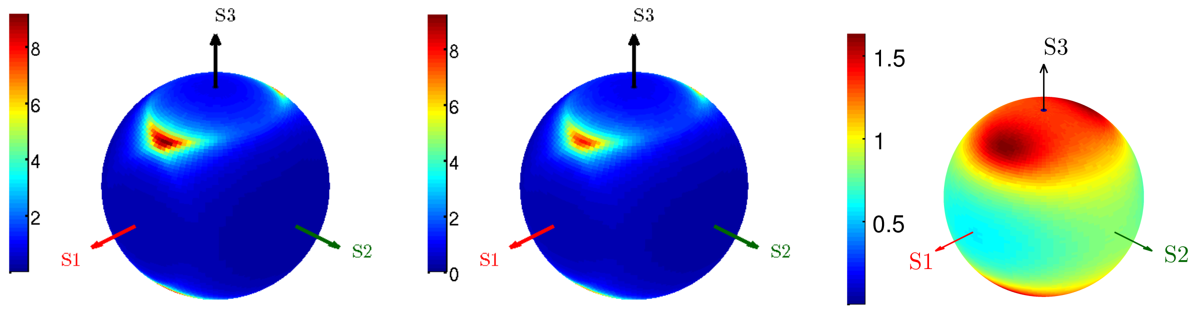
$$\chi_{\boldsymbol{\tau}_\phi} = -2 \boldsymbol{\tau}_\phi \cdot \nabla \phi \quad (3)$$

Given the critical role of $\boldsymbol{\tau}_\phi$ orientation in regulating subfilter dissipation (Eq. 3), we investigate the vector's alignment behavior in homogeneous isotropic turbulence, and its relation to the strain-rate tensor. More precisely, the goals are three-fold: 1) to relate the predicted vector's orientation to known velocity-dependent quantities; 2) to identify the influence (if any) of the Schmidt number of the transported scalar, and the influence of width and shape of the filtering kernel used; and 3) to assess the ability of various existing models in predicting this orientation.

VECTOR-ORIENTATION AND THE STRAIN-RATE EIGENFRAME

The orthonormal reference frame formed by the eigenvectors of the local strain-rate tensor ($S_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$) presents a suitable choice for examining vector-orientation of various quantities of interest. This is due to the fact that the dynamics of passive scalar transport (as well as those of the velocity field itself) are governed largely by straining from the velocity field. To inspect the alignment of $\boldsymbol{\tau}_\phi$, the appropriate terms in Eq. 2 are obtained by filtering DNS datasets spatially. Three different filtering kernels, namely the Gaussian filter, the box (or top-hat) filter, and the sharp-spectral filter are used here. Figure 1 shows the three-Dimensional (3D) joint Probability Density Functions (PDF) of $\boldsymbol{\tau}_\phi$ in the strain-rate eigenframe, computed pointwise for $Re_\lambda = 30,\ Sc = 256$. The panels show the alignment behavior for the three different filters, corresponding to an effective filter-width where approximately 93% ($\kappa_c \eta_B = 0.1$) of the scalar modes have been filtered out. At these cutoff widths, $\boldsymbol{\tau}_\phi$ exhibits strong preference for alignment at a particular angle in the $S_1 - S_3$ plane when using the Gaussian and Box filters. The reason for this behavior of $\boldsymbol{\tau}_\phi$ orientation, as well as that for the observed independence from the Schmidt number [3], are explained using the leading order term in the Taylor-series expansion of Eq. 2. More precisely, the expansion takes the following form for a variety of filtering kernels, including the Gaussian and box filters [2]:

$$\begin{aligned} \boldsymbol{\tau}_\phi &= \widetilde{\boldsymbol{u}\phi} - \widetilde{\boldsymbol{u}} \widetilde{\phi} \\ &= C\, \nabla \widetilde{\boldsymbol{u}} \cdot \nabla \widetilde{\phi} + O(\Delta^4) & (4) \end{aligned}$$



(a) Gaussian (b) Box (c) Sharp-spectral

Figure 1. Alignment of the subfilter flux vector ($\boldsymbol{\tau}_\phi$) in the filtered strain-rate ($\widetilde{S}_{ij}$) eigenframe. S_1 represents the direction of the most extensive eigenvector of S_{ij} (positive eigenvalue: $\lambda_1 > 0$), S_2 the intermediate one, and S_3 the most compressive one ($\lambda_3 < 0$).

The leading order term in Eq. 4 constitutes the tensor-diffusivity model. The coefficient C depends on the second moment ($\int_{-\infty}^{\infty} x^2 G(x) dx$) of the filtering kernel, and is identical for the forms of the Gaussian and box filtering kernels ($\Delta^2/12$). This implies that, to leading order, $\boldsymbol{\tau}_\phi$ is virtually identical for both the Gaussian and box filters, which can be confirmed from Figs. 1(a) and 1(b). We note that moment integrals of the sharp-spectral kernel do not converge to finite values for order greater than 0. The consequence of using the sharp filter is discussed in greater detail in [3]. The velocity gradient tensor ($\nabla \boldsymbol{u}$) in Eq. 4 is decomposed into its symmetric and anti-symmetric parts, *i.e.*, the strain-rate tensor $S_{ij} = 1/2(u_{i,j} + u_{j,i})$ and the rotation tensor $\Omega_{ij} = 1/2(u_{i,j} - u_{j,i})$, leading to:

$$\boldsymbol{\tau}_\phi \quad = \quad \frac{\Delta^2}{12} \boldsymbol{S} \cdot \nabla \widetilde{\phi} + \frac{\Delta^2}{24} \boldsymbol{\omega} \times \nabla \widetilde{\phi} \tag{5}$$

$\nabla \widetilde{\phi}$ shows strong preference for orientation along the S_3 direction (or $\hat{\boldsymbol{e}}_3$ unit vector), whereas $\boldsymbol{\omega}$ aligns with $\hat{\boldsymbol{e}}_2$ [3]. Thus, we can make the following simplifications:

$$\boldsymbol{\tau}_\phi \quad \approx \quad \frac{\Delta^2}{12} \left(\pm \boldsymbol{S} \cdot |\nabla \widetilde{\phi}| \, \widehat{e}_3 \pm \frac{1}{2} |\boldsymbol{\omega}| \, \widehat{e}_2 \times |\nabla \widetilde{\phi}| \, \widehat{e}_3 \right) \tag{6}$$

$$\approx \quad \pm \frac{\Delta^2}{12} |\nabla \widetilde{\phi}| \left(\lambda_3 \, \widehat{e}_3 \pm \frac{1}{2} |\boldsymbol{\omega}| \, \widehat{e}_1 \right) \tag{7}$$

As is evident from Eq. 7, $\boldsymbol{\tau}_\phi$ has measurable components only in the S_1 and S_3 directions. This explains the strong aversion to alignment in the S_2 direction (Fig. 1). Equation 7 can further be used to extract deterministic information about $\boldsymbol{\tau}_\phi$ orientation in the $S_1 - S_3$ plane. The tensor-diffusivity model-predicted SFF vector (Eq. 7) makes an angle of

$$\theta_{S_3} = \arctan \left(\pm \frac{|\boldsymbol{\omega}|}{2\lambda_3} \right) \tag{8}$$

with the S_3 axis, whereas the exact $\boldsymbol{\tau}_\phi$ (Eq. 2) subtends the following angle:

$$\alpha_{S_3} = \arccos \left(\frac{\boldsymbol{\tau}_\phi \cdot \widehat{e}_3}{|\boldsymbol{\tau}_\phi| \, |\widehat{e}_3|} \right). \tag{9}$$

The PDFs of θ_{S_3} and α_{S_3} match quite well [3]. The complete independence of the angle in Eq. 8 from any scalar field related variable explains the Schmidt number and filter-width independence of the orientation of $\boldsymbol{\tau}_\phi$ observed in [3]. However, the magnitude of $\boldsymbol{\tau}_\phi$ *will* depend on the filtered scalar field, as is clearly evident from the presence of $|\nabla \widetilde{\phi}|$ in Eq. 7. The observations in this section confirm that the tensor-diffusivity model provides an excellent means of representing the subfilter-scale scalar-flux vector orientation analytically, when using the Gaussian or box filters.

References

[1] A. Leonard. Energy cascade in large-eddy simulations of turbulent fluid flows. *Adv. Geophys* **18 A**: 237–248, 1975.
[2] D. Carati, G.S. Winckelmans, H. Jeanmart, ON the modelling of the subgrid-scale and filtered-scale stress in large-eddy simulation. *J. Fluid Mech* **441**: 119-138, 2001.
[3] S. Verma and G. Blanquart. Subfilter scalar-flux vector orientation in homogeneous isotropic turbulence. *Phys. Rev. E* **89**: 063015, 2014.