# NEW EXACT COHERENT STATES IN CHANNEL FLOW

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*Abstract* Three spatially periodic travelling wave exact coherent states are presented for channel flow. Two of the flows, which are asymmetric with respect to the channel centreplane, are derived by homotopy from solutions for channel flow subject to a spanwise rotation investigated by [1]. The third flow satisfies a half-turn rotational symmetry about a point on the channel centreplane, and turns out to be the flow from which one of the asymmetric flows bifurcates in a symmetry breaking bifurcation. One of the asymmetric flows is found to substantially reduce the value of the lowest Reynolds number at which exact solutions are known to exist down to 665.

## INTRODUCTION

During the past few decades, significant insights into the transition to turbulence have been gained by the identification of invariant solutions, called 'exact coherent states', to the governing Navier-Stokes equations. For channel flow the known solutions remain relatively few [2,3,4], and so the objective of the present study is to seek to identify new exact solutions with the aim of enhancing understanding of transition and turbulence in this canonical flow.

## MODEL

The present new solutions originate from travelling wave flows in channel flow rotating at a constant rate $\Omega^*$ about a spanwise axis. Such a flow is governed by the dimensionless Navier-Stokes equations and incompressibility condition

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \boldsymbol{\Omega}\times\boldsymbol{u} = -\nabla p + \nabla^2\boldsymbol{u}, \qquad \nabla\cdot\boldsymbol{u} = 0, \tag{1}$$

expressed in the rotating frame of reference, where velocity $\boldsymbol{u}$ has components $(u, v, w)$ in the streamwise $(x)$, spanwise $(y)$ and wall-normal $(z)$ directions respectively, $t$ denotes time and $p$ a modified pressure that includes the centrifugal force term. All lengths have been scaled by half the channel width $L$, while velocities have been scaled by $V = \nu/L$, where $\nu$ denotes the kinematic viscosity, and the rotation number $\Omega = 2\Omega^* L^2/\nu$ has been introduced where $\boldsymbol{\Omega} = [0, \Omega, 0]$. A laminar basic flow solution of (1) subject to no-slip at the channel wall, $\boldsymbol{u}(z = \pm 1) = 0$, is given by $\boldsymbol{u} = (u_0(z), 0, 0)$, $p = p_0(x, z)$ in which $u_0(z) = R(1 - z^2)$ and $p_0 = -2Rx + \Omega R(z - z^3/3)$, where $R = L^3 J/2\rho\nu^2$ denotes the Reynolds number for constant density $\rho$ and a constant imposed pressure gradient in the streamwise direction $-J$, $(J > 0)$.

## NUMERICAL METHODS

We are interested in the development of disturbances which are superimposed on the laminar basic flow solution $(u_0, p_0)$. A typical component, $q$, of the disturbances is expressed in the following travelling wave form

$$q(x, y, z, t) = \sum_{l=-L}^{L}\sum_{m=-M}^{M}\sum_{n=-N}^{N} q_{lmn}(t)\exp[im\alpha(x - ct) + in\beta y]T_l(z), \tag{2}$$

where $\alpha$ and $\beta$ are the wavenumbers in the streamwise and the spanwise directions, respectively, $c$ is the wavespeed and $T_l(z)$ denote Chebyshev polynomials modified to satisfy the no-slip condition. Upon discretizing the equations by a collocation method, the resulting algebraic equations for the $q_{lmn}$ amplitudes and $c$ are solved by Newton's method.

## RESULTS

With the present homotopy strategy, of the 15 distinct travelling wave flows of [1] (labelled $\mathscr{G}_1, \cdots, \mathscr{G}_{15}$ therein) in rotating channel flow, only two, referred to here as the TW1 flow and the TW2 flow, originating from the $\mathscr{G}_1$ tertiary flow and the $\mathscr{G}_{13}$ secondary flow, respectively, were able to be continued to yield non-rotating channel flow solutions, as seen in Figure 1(a). The wavenumbers of the TW1 and TW2 flows marked by the points A, B, C, D in Figure 1(a) are optimized in order to identify the minimum value of $R$ at which these solutions first appear (see Figure 1(b)). It should be noted that the TW2 flow first appears in a saddle-node bifurcation at $R = 665.3$ for $(\alpha, \beta) = (1.32, 2.89)$.

The TW1 flow features one strong low-speed streak per spanwise wavelength in the streamwise component of velocity, while the TW2 flow features two such low-speed streaks. In both cases the streaks are sinusoidal and are flanked by staggered vortex structures (see Figure 2 (a) and (b)). Both of these flows are asymmetric about the channel centreplane. A third flow, TW3, from which the TW1 flow bifurcates in a symmetry-breaking bifurcation, is additionally found (see

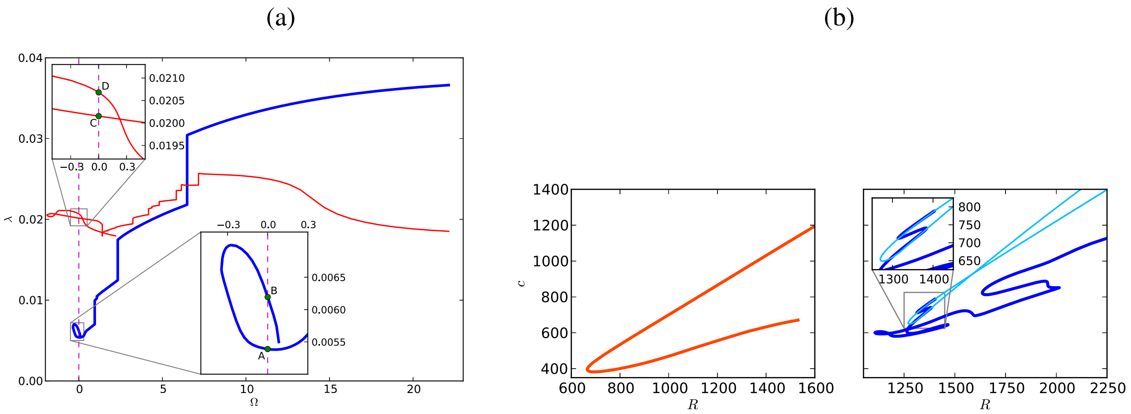


**Figure 1.** (a): Friction factor $\lambda$ against $\Omega$ with thick blue lines showing the homotopy path from the $\mathscr{G}_1$ rotating flow solution to the TW1 channel flow solution, and the thin red lines showing the path from the $\mathscr{G}_{13}$ rotating flow solution to the TW2 channel flow solution. (b): Wavespeed $c$ against $R$ for (left) TW2 and (right) TW1 (thick dark blue line) channel flows obtained for the optimum wavenumber parameters $(\alpha, \beta) = (1.32, 2.89)$ (TW2) and ((1.63,3.25) (TW1). The right-hand plot also shows the flow TW3 (thin light-blue line), from which the asymmetric TW1 flow bifurcates at $R = 1403$.

Figure 1 (b)). The TW3 flow satisfies a half-turn rotational symmetry in the spanwise - wall normal plane about a point on the channel centreplane, as shown in Figure 2 (c).

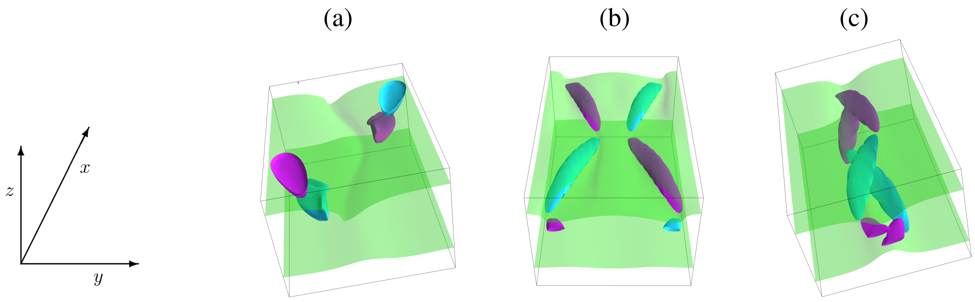


**Figure 2.** The upper-branch flow structures of (a) TW1 at $(R, \alpha, \beta) = (1750, 1.63, 3.25)$, (b) TW2 at $(R, \alpha, \beta) = (1200, 1.32, 2.89)$ and (c) TW3 at $(R, \alpha, \beta) = (1500, 1.63, 3.25)$. The TW1 and TW3 flows feature one low-speed streak per spanwise wavelength while the TW2 flow features two such streaks. The TW1 and TW2 flows are asymmetric about the channel centreplane while the TW3 flow satisfies a half-turn symmetry about a point in this plane. The blue (purple) isosurfaces show 0.7 of the minimum (maximum) of the streamwise component of vorticity, while the translucent sheets show isosurfaces of 0.5 of the maximum of $u$.

## SUMMARY

In the present study three new exact coherent states in channel flows have been obtained by continuing solutions obtained for channel flow subject to a system rotation about a spanwise axis to non-rotating channel flow by homotopy. The TW1 flow can perhaps be regarded as an asymmetric (with respect to the channel centreplane) version of the channel flow solution found by [2], since it shares all the main features of the latter flow other than symmetry about this plane. The TW2 flow more closely resembles the MS-A flow recently presented by [3], but appears to be distinct to this latter flow since the minimum Reynolds numbers at which these two flows first appear, and the corresponding optimum wavenumbers, are markedly different. In particular, the present TW2 flow reduces the lowest known Reynolds number at which exact solutions are known to exist down to 665, from the previous known minimum of 805.5 reported by [3]. The TW3 flow instead satisfies a half-turn rotational symmetry, and yields the TW1 solution in a symmetry breaking bifurcation.

### References

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