ASSESSING LATE-TIME SINGULAR BEHAVIOUR IN SYMMETRY-PLANE MODELS OF 3D EULER FLOW

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Abstract Motivated by work on stagnation-point type exact solutions of the 3D Euler fluid equations by Gibbon [Gibbon *et. al.* Phys. D, 132, 497, (1999)] and the subsequent demonstration of finite-time blowup by Constantin [Constantin, Math. Res. Notices, 9, 455, (2000)] we introduce a one-parameter family of models of the 3D Euler equations on a 2D symmetry plane. These models provide a collection of blow-up scenarios which admit analytical solutions and are computationally inexpensive in comparison to the full 3D Euler equations. We take advantage of these features to examine the efficacy of novel methods which aid the assessment of finite-time blow-up in numerical simulations. The principal of these is the mapping to regular systems [Bustamante, Phys. D, 240, 1092, (2011)]; a bijective nonlinear mapping of time and the prognostic variables based on a Beale-Kato-Majda (BKM) type supremum norm regularity condition [Beale *et. al.* Commun. Math. Phys. 94, 61, (1984)]. We show a 3 order of magnitude increase of accuracy of the singularity time when employing the mapping with negligible additional computational expense. An investigation of the spectra of the primary field (vortex stretching rate) allows us to confirm a power law decrement of the analyticity-strip width with time in agreement with rigorous bounds bridging between the global spatial behaviour and BKM theorems [Bustamante & Brachet, Phys. Rev. E. 86, (2012)].

SYMMETRY PLANE MODEL AND MAPPING TO REGULAR SYSTEMS

We define our model equations by considering the 3D Euler equations on a symmetry plane at $z = 0$ where velocity is parallel to the plane and vorticity perpendicular. The dynamics on the plane can be described by equations for two scalar fields, vorticity, ω and vortex stretching rate γ. In general these equations require some closure as there is still dependence of the full 3D pressure. Gibbon [5] considered exact solutions of 3D Euler (albeit with infinite energy) where the pressure ansatz, $p_{zz} = -2\langle\gamma^2\rangle$, ensures horizontal periodicity. Here we extend this model via a single-parameter family of closures which allows for some spatial dependence of p_{zz} on the plane and therefore dynamics closer to full 3D Euler. The equations are

$$\frac{\partial\gamma}{\partial t} + \mathbf{u}_\mathrm{h}\cdot\nabla_\mathrm{h}\gamma = (2+\lambda)\langle\gamma^2\rangle - (1+\lambda)\gamma^2\,, \qquad (1)$$

$$\frac{\partial\omega}{\partial t} + \mathbf{u}_\mathrm{h}\cdot\nabla_\mathrm{h}\omega = \gamma\omega\,, \qquad (2)$$

where $(x, y) \in [0, 2\pi] \times [0, 2\pi]$, $\lambda \in \mathbb{R}$ is a free parameter and subscript h denotes horizontal part. The case $\lambda = 0$ recovers the equations of [5]. Analytic solutions of the above system can be obtained for supremum norm quantities for specified initial conditions by solving along characteristics. The parameter space is partitioned into regions where the behaviour is singular or non-singular, controlled by a classical Beale-Kato-Majda type [1] theorem: $\int_0^T \|\gamma(\cdot,t)\|_\infty < \infty$ for bounded solutions.

Bustamante [2] presented a method for bijectively mapping systems whose regularity is governed by BKM-type supremum norms to one which is globally regular in time. Such a mapping removes some of the ambiguity of numerical assessment of singular behaviour and can increase the accuracy of important quantities. The BKM theorem provides a new 'mapped' time and a rescaling to mapped variables

$$\tau(t) = \int_0^t \|\gamma(\cdot,t')\|_\infty \mathrm{d}t', \qquad \gamma_\mathrm{map}(x,y,\tau) = \frac{\gamma(x,y,t)}{\|\gamma(\cdot,t)\|_\infty}\,, \qquad \omega_\mathrm{map}(x,y,\tau) = \frac{\omega(x,y,t)}{\|\gamma(\cdot,t)\|_\infty}\,, \qquad (3)$$

resulting in the mapped version of system (1)-(2)

$$\frac{\partial\gamma_\mathrm{map}}{\partial\tau} + \mathbf{u}_\mathrm{map}\cdot\nabla\gamma_\mathrm{map} = (2+\lambda)\langle\gamma^2_\mathrm{map}\rangle - (1+\lambda)\gamma^2_\mathrm{map} + \sigma_\infty\,\gamma_\mathrm{map}\left\{1+\lambda-(2+\lambda)\langle\gamma^2_\mathrm{map}\rangle\right\} \qquad (4)$$

$$\frac{\partial\omega_\mathrm{map}}{\partial\tau} + \mathbf{u}_\mathrm{map}\cdot\nabla\omega_\mathrm{map} = \gamma_\mathrm{map}\,\omega_\mathrm{map} + \sigma_\infty\,\omega_\mathrm{map}\left\{1+\lambda-(2+\lambda)\langle\gamma^2_\mathrm{map}\rangle\right\}. \qquad (5)$$

where σ_∞ is the sign of the extremum of γ. The numerical solution therefore requires accurate computation of $\|\gamma(\cdot,t)\|_\infty$ and normalisation of the fields. In this way the fields remain bounded and singularity time T^* is mapped to $\tau = \infty$. Recovery of the original variables involves solving the evolution ODE for $\|\gamma(\cdot,t)\|_\infty$, resulting in the following integration:

$$||\gamma(\cdot,t(\tau))||_\infty = \|\gamma_0\|_\infty \exp\left[-(1+\lambda)\int_0^\tau \sigma_\infty\,\mathrm{d}\tau' + (2+\lambda)\int_0^\tau \sigma_\infty\langle\gamma^2_\mathrm{map}\rangle\mathrm{d}\tau'\right]. \qquad (6)$$

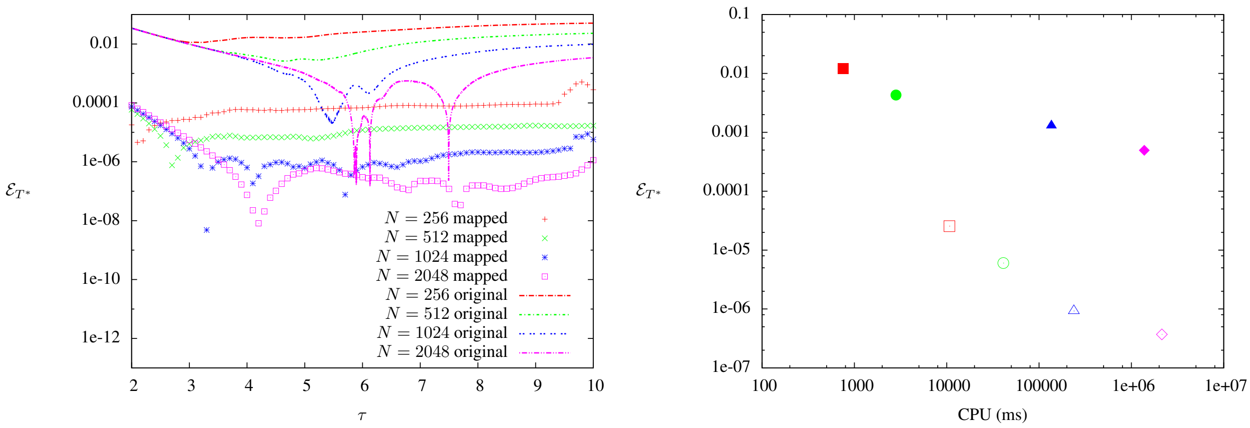

Figure 1. Left: errors in the running estimates of singularity time T^* using data from original system's numerical integration (lines) and mapped system's numerical integration (symbols) at different resolutions. Right: Best guess error for T^* in the original system (filled symbols) and the mapped system (open symbols) at resolutions: $N = 256$ (red squares), $N = 512$ (green circles), $N = 1024$ (blue triangles), $N = 2048$ (magenta diamonds). The CPU overhead of applying the mapping is shown to be more than covered by the accuracy gain.

DIAGNOSIS OF SINGULARITY

As numerical simulations are the primary method of investigating possible singularities of fluid equations it is important we develop accurate and reliable means of diagnosing these extreme events. Here we solve both systems using the same pseudospectral method and compare typical diagnostics with the aim of identifying both the benefits of the mapping *and* the appropriateness of our symmetry plane models. In the results here $\lambda = -1.5$ produces finite-time singularity and suitably chosen initial data gives straightforward expressions in the analytic solution of the supremums.

One key measure is the singularity time T^*. From the mapped system this can be estimated by integrating the reciprocal of equation (6) in τ, up to some suitable value and then correcting the remainder of the integration using a two parameter fit for $\langle \gamma^2_{\text{map}} \rangle$. In the original system T^* can be estimated by again using a two parameter fit, this time for the late-time asymptote of $\|\gamma(\cdot, t)\|_\infty$ using a power-law ansatz. The analytic solution provides an exact value for T^* allowing a relative error to be defined. Figure 1 shows running calculations of this relative error at various resolutions in both mapped and original systems demonstrating that the mapped system increases accuracy by around 3 orders of magnitude. Also shown is the 'best guess' error for each resolution against CPU time showing that the accuracy gain observed in the mapped system comes with only a small additional overhead associated with determining $\|\gamma(\cdot, t)\|_\infty$, its position and normalising the fields.

Other benefits of the mapping can be established when examining the late time behaviour of the spectrum of γ and the decay of the analyticity strip width. In particular in the $\lambda = 0$ case studied by [5, 7] we find that the true converged, asymptotic regime lies beyond the floating point precision barrier in the original t variables, leaving it inaccessible to simulations of the original system even when employing adaptive time stepping.

Details can be found in a preprint here [6] and further manuscripts to follow.

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