LOG-STABLE LAW OF ENERGY DISSIPATION RATE FOR TURBULENCE INTERMITTENCY

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Abstract To describe the small-scale intermittency of turbulence, a self-similarity is imposed on the probability density function of a logarithm of the rate of energy dissipation smoothed over a length scale among those in the inertial range. The result is an extension of Kolmogorov's classical theory in 1941, i.e., a one-parameter framework where the logarithm obeys some stable distribution. We obtain the scaling laws for the dissipation rate and for the two-point velocity difference.

INTRODUCTION

Turbulence is intermittent at small length scales in the inertial and dissipation ranges. A representative example is the rate of energy dissipation per unit mass ε. If smoothed over a scale in the inertial or dissipation range, it is significant only within a fraction of the space. This fraction decreases with a decrease in the scale. Such small-scale intermittency has been a subject of intensive studies for decades [1].

The central issue is to obtain a statistical framework. Especially for the dissipation rate ε_r smoothed over length scale r in the inertial range, we expect a power-law scaling $\langle \varepsilon_r^m \rangle \propto r^{\tau_m}$. The exponent τ_m has to satisfy $\tau_m/m > -D$ at $m > 0$ [1] and $\tau_m/m \to -D$ as $m \to +\infty$ [2], where D is the dimension of the smoothing region. We are to formulate a framework that is consistent with such constraints [3].

BASIC SETTINGS

For a fully developed state of homogeneous and isotropic turbulence, the dissipation rate ε is smoothed over length scales r in the inertial range. We consider the one-dimensional smoothing case $D = 1$, by averaging ε over a segment of length r centered at a position x along a line in the three-dimensional space as $\varepsilon_r(x) = (1/r) \int_{|x'-x| \le r/2} \varepsilon(x') dx'$. The three-dimensional smoothing case $D = 3$ is also considered, by averaging ε over a spherical volume of diameter r centered at a position $\boldsymbol{x}$ as $\varepsilon_r(\boldsymbol{x}) = (6/\pi r^3) \int_{|\boldsymbol{x}'-\boldsymbol{x}| \le r/2} \varepsilon(\boldsymbol{x}') d\boldsymbol{x}'$. By using the largest scale R of the inertial range, we define random variables $\chi_r(x) = \ln[r\varepsilon_r(x)/R\varepsilon_R(x)]$ for $D = 1$ and $\chi_r(\boldsymbol{x}) = \ln[r^3\varepsilon_r(\boldsymbol{x})/R^3\varepsilon_R(\boldsymbol{x})]$ for $D = 3$.

DISSIPATION RATE

The following conditions are imposed on the inertial range at $r \le R$: (i) the probability density function (PDF) of $\chi_{r_1} - \chi_{r_2}$ depends only on r_1/r_2 for any pair of r_1 and r_2; (ii) the PDFs of $\chi_{r_1} - \chi_{r_2}$, $\chi_{r_2} - \chi_{r_3}$, ..., $\chi_{r_{N-1}} - \chi_{r_N}$ do not depend on one another for any series of $r_1 < r_2 < \cdots < r_N$; and (iii) the PDF of χ_r is self-similar. If the sign $\triangleq$ is used to denote that the two random variables obey the same distribution, the condition (iii) is such that a constant $C_{r_1,r_2} > 0$ exists for any pair of r_1 and r_2 to have $C_{r_1,r_2}\chi_{r_1} \triangleq \chi_{r_2}$. From these conditions, it follows that the variable χ_r is strictly stable at each scale r and is described as $\chi_r \triangleq \left[\ln\left(R^\beta/r^\beta\right)\right]^{1/\alpha} \chi_*$ with $0 < \alpha \le 2$ and $\beta > 0$.

Here is a mathematical explanation [4]. The conditions (i) and (ii) allow us to regard χ_r as a stochastic Lévy process χ_t for the time parameter $t = \ln(R^\beta/r^\beta) \ge 0$. If χ_t has a self-similar PDF, χ_t is said to be strictly stable. By defining χ_* as χ_t at $t = 1$, we have $\chi_t \triangleq t^{1/\alpha}\chi_*$ with a parameter $0 < \alpha \le 2$.

The strictly stable distributions make up a three-parameter family. Except for $\alpha = 1$, the characteristic function of χ_* is $\langle \exp(i\chi_*\xi) \rangle = \exp(-\lambda|\xi|^\alpha e^{i\pi\theta\xi/2|\xi|})$. While $\lambda > 0$ determines the width of the PDF, α and θ determine its shape ($|\theta| \le \alpha$ for $0 < \alpha < 1$ and $|\theta| \le 2 - \alpha$ for $1 < \alpha \le 2$). We focus on the distributions for $0 < \alpha < 1$ and $\theta = \alpha$. They alone are totally skewed to the left, i.e., $\chi_* \le 0$, which is required from $\chi_r \le 0$.

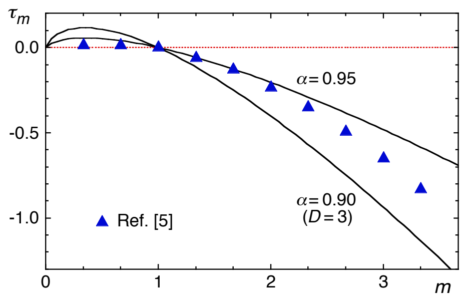
To formulate the case of $D = 1$, we use $\langle r^m\varepsilon_r^m/R^m\varepsilon_R^m \rangle = \langle \exp(m\chi_r) \rangle$, which is rewritten by using the above relations and by replacing ξ with $-im$. This replacement holds valid at $m \ge 0$. The result is $\langle \varepsilon_r^m/\varepsilon_R^m \rangle = (r/R)^{-m+m^\alpha\beta\lambda\exp(\pi\alpha/2)}$. Finally, to eliminate the parameters β and λ, we impose a condition (iv) such that ε_R is constant at $\langle \varepsilon \rangle$. Then,

$$\frac{\langle \varepsilon_r^m \rangle}{\langle \varepsilon \rangle^m} = (r/R)^{-m+m^\alpha} \quad \text{with } \ 0 < \alpha < 1 \ \text{ for } \ D = 1 \ \text{ at } \ m \ge 0. \tag{1a}$$

By replacing the length ratio r/R with the volume ratio r^3/R^3, the case of $D = 3$ is formulated as

$$\frac{\langle \varepsilon_r^m \rangle}{\langle \varepsilon \rangle^m} = (r/R)^{-3m+3m^\alpha} \quad \text{with } \ 0 < \alpha < 1 \ \text{ for } \ D = 3 \ \text{ at } \ m \ge 0. \tag{1b}$$

Thus, we have $\tau_m = -Dm + Dm^\alpha$ for $\langle \varepsilon_r^m \rangle \propto r^{\tau_m}$ [3]. The intermittency is described by the parameter α. In particular, the limit $\alpha \to 1$ reproduces the 1941 theory of Kolmogorov, i.e., $\tau_m = 0$ [1].



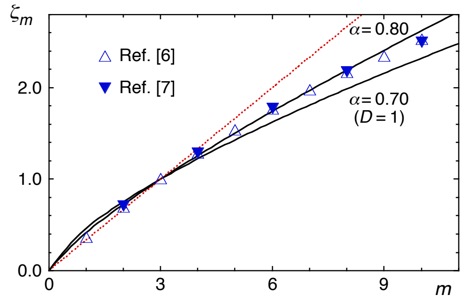


Figure 1. Comparison of our framework with numerical simulations [5, 6, 7]. The dissipation exponent τ_m is for the case of $D = 3$ in Eq. (1b). The velocity exponent ζ_m is for the case of $D = 1$ in Eq. (2). Dotted lines denote the 1941 theory of Kolmogorov [1].

VELOCITY DIFFERENCE

Also for the two-point velocity difference $\delta u_r(x) = u(x + r) - u(x)$, where u is parallel to the line through the two points, we have a power-law scaling $\langle|\delta u_r^m|\rangle \propto r^{\zeta_m}$ at $m \geq 0$ [3]. The exponent ζ_m is from the exponent τ_m for the dissipation rate ε_r via $\langle|\delta u_r^m|\rangle \propto \langle(r\varepsilon_r)^{m/3}\rangle$ and hence via $\zeta_m = \tau_{m/3} + m/3$ [1]. In the case of the one-dimensional smoothing $D = 1$ [Eq. (1a)],

$$\langle|\delta u_r^m|\rangle \propto r^{(m/3)^\alpha} \quad \text{with} \ \ 0 < \alpha < 1 \ \ \text{at} \ \ m \geq 0. \tag{2}$$

The limit $\alpha \to 1$ reproduces the 1941 theory of Kolmogorov, i.e., $\zeta_m = m/3$ [1]. We do not consider the three-dimensional smoothing case $D = 3$ [Eq. (1b)]. The velocity difference δu_r is an integral of the velocity derivative $\partial u/\partial x$ over the separation r, while an integral of $(\partial u/\partial x)^2$ is one of the components of the dissipation rate ε_r. Regions of these two integrations are identical in the case of $D = 1$. They are not identical in the case of $D = 3$.

COMPARISON WITH SIMULATIONS

Figure 1 compares the scaling laws $\langle\varepsilon_r^m\rangle \propto r^{\tau_m}$ and $\langle|\delta u_r^m|\rangle \propto r^{\zeta_m}$ of our framework with those observed in numerical simulations of forced steady states of homogeneous and isotropic turbulence [5, 6, 7]. Since the simulations agree well with our framework, its conditions (i)–(iv) should serve at least as a good approximation of the actual turbulence.

CONCLUDING COMMENTS

The present framework is necessary and sufficient for the self-similar PDF of $\ln \varepsilon_r$. We expect such a self-similarity from the local and instantaneous transfer of the kinetic energy, which occurs not only to the smaller scales but also to the larger scales. These scales interact with one another and should have settled into some self-similar state.

The present framework is based on that of Kida [8], which gives $\tau_m = -\mu(m^\alpha - m)/(2^\alpha - 2)$ with two free parameters $\mu > 0$ and $0 < \alpha \leq 2$ even if the smoothing dimension D is given. We have obtained $\tau_m = -Dm + Dm^\alpha$ with $0 < \alpha < 1$ that always satisfies $\tau_m/m > -D$ at $m > 0$ and $\tau_m/m \to -D$ as $m \to +\infty$.

We have not considered the fluctuations at the largest scale R of the inertial range. In fact, ε_R fluctuates significantly even if the turbulence is fully developed and is filling in the space [9, 10]. Although such fluctuations are unlikely to affect the value of the exponent τ_m [1], we would have to incorporate them into the present framework in the future.

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